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Numerical analysis

Numerical scheme for multilayer shallow-water model in the low-Froude number regime ☆

Schéma numérique pour les modèles de Saint-Venant multi-couche à faible nombre de Froude

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ABSTRACT

The aim of this note is to present a multi-dimensional numerical scheme approximating the solutions to the multilayer shallow-water model in the low-Froude-number regime. The proposed strategy is based on a regularized model where the advection velocity is modified with a pressure gradient in both mass and momentum equations. The numerical solution satisfies the dissipation of energy, which acts for mathematical entropy, and the main physical properties required for simulations within oceanic flows.

RÉSUMÉ

Le but de cette note est de présenter un schéma numérique multi-dimensionnel rapprochant les solutions du modèle de Saint-Venant multi-couche en régime de faible nombre de Froude. La stratégie proposée est basée sur un modèle régularisé où la vitesse de transport est modifiée par un gradient de pression dans les équations de la masse et de la quantité de mouvement. La solution numérique satisfait la dissipation d'énergie, jouant le rôle de l'entropie du point de vue mathématique, et les principales propriétés physiques nécessaires aux simulations dans le cadre des écoulements océaniques.

1. Introduction

The current note is devoted to the numerical resolution of the multilayer shallow-water model. Let us consider a set of L immiscible, homogeneous, inviscid, and incompressible superposed fluids with free surface and without surface tension. The pressure is assumed to be hydrostatic and constant at the free surface. In addition, we supposed that the vertical acceleration is small enough to make the flow satisfy the shallow-water assumption. The ith layer of fluids has a constant

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density ρ_i, a thickness $h_i(t,x)$ and a depth-averaged horizontal velocity $v_i(t,x)$, with $t \geq 0$ standing for the time variable and $x = (x_1, x_2) \in \mathbb{R}^2$ the horizontal position. The governing equations are given by the multilayer shallow-water equations, i.e.

$$(SW) \quad \begin{cases} \partial_t h_i + \nabla_x \cdot (h_i v_i) = 0, \\ \partial_t (h_i v_i) + \nabla_x \cdot (h_i v_i \otimes v_i) + \dfrac{h_i}{\rho_i} \nabla_x p_i = 0, \end{cases} \tag{1}$$

with g is the gravitational acceleration and the pressure being given by the hydrostatic relation, i.e. $p_i = g \sum_{j=1}^{L} \rho_{\min(i,j)} h_j$, with the layer numbered downwards from the free surface. The multi-layer shallow-water model (1) could be obtained from Euler equations by vertical averaging across the layer depth, see [2]. Previous studies presented the criterion of well-posedness for the multilayer shallow-water model (1), see [7,11]. In particular, the layers should be organized with the heaviest at the bottom to the lightest at the top, i.e. $\rho_1 < \rho_2 < \cdots < \rho_{L-1} < \rho_L$. In addition, the vertical variation of velocity should be small enough in comparison to the layer thickness, more precisely a required condition for the hyperbolicity of (1) reads:

$$\frac{\|v_{i+1} - v_i\|^2}{g(h_{i+1} + h_i)} < 1 - \frac{\rho_i}{\rho_{i+1}}. \tag{2}$$

The main features of the multilayer shallow-water model are its mathematical structure, more precisely hyperbolic for reasonable regimes, and the underlying conservative laws. More precisely, the multilayer shallow-water model preserves the water volume of each layer (first equation of (1)), the momentum of the column of water

$$\partial_t \sum_{i=1}^{L} (\rho_i h_i v_i) + \nabla_x \cdot \sum_{i=1}^{L} (\rho_i h_i v_i \otimes v_i) + \nabla_x \mathcal{E} = 0 \tag{3}$$

and the total energy $E = \mathcal{E} + \mathcal{K}$ (for smooth solutions)

$$\partial_t E + \nabla_x \cdot \sum_{i=1}^{L} \big((2\mathcal{E}_i + \mathcal{K}_i) v_i \big) = 0 \tag{4}$$

where the potential energy is defined by $\mathcal{E} = \sum_{i=1}^{L} \mathcal{E}_i$ with $\mathcal{E}_i = \frac{1}{2} h_i p_i$ and the kinetic energy is defined by $\mathcal{K} = \sum_{i=1}^{L} \mathcal{K}_i$ with $\mathcal{K}_i = \frac{1}{2} \rho_i h_i \|v_i\|^2$.

In the current work, we are interesting in the application of the multilayer shallow-water model to describe an oceanic environment characterized by a weakly stratified density, i.e. $0 < 1 - \frac{\rho_i}{\rho_{i+1}} \ll 1$, see [9]. The multilayer shallow-water model could be used to describe weakly stratified flows assuming that the Froude number is small enough in a suitable referential such that the required condition (2) holds. Classical numerical schemes used to approximate shallow-water models, like Riemann solvers, require the estimation, or at least an upper bound, of the eigenvalues. In [5] and [1], the authors present a numerical strategy based on the estimation of the eigenvalues realized in [10]. This estimation is valid for weakly stratified density and in the two-layer case. It is hardly adaptable to an arbitrary number of layers and for all stratifications. In [3] and [4], the authors present a strategy to adapt the resolution of the mono-layer case, currently well known, to the multi-layer one. However, these solvers are well known to be too much dissipative in the low-Froude-number regime, and lead to restrictive CFL conditions in the context of oceanic flow.

To overcome these drawbacks, we adapt the numerical scheme presented in [8] for multiphasic Euler equations in the low-Mach-number regime. This strategy does not require the estimation of eigenvalues and is able to recover low-Mach number flows. The multilayer shallow-water model introduces new difficulties compared to multiphasic Euler equations. More precisely, the number of unknowns is larger since the velocities of each layer are not the same and the pressure term is not conservative in the multilayer shallow-water model.

2. Regularized model

As it is shown in [6], the low-Froude-number regime, by analogy with the low-Mach-number regime, requires the centered discretization of the pressure term. However, this discretization introduces a numerical source term of the discrete energy that leads to an unstable solution. To overcome this difficulty, the key point is to modify the convective discharge in both mass conservation and momentum balance such that the regularized model yields:

$$(SW_\varepsilon) \quad \begin{cases} \partial_t h_i^\varepsilon + \nabla_x \cdot q_i^\varepsilon = 0, \\ \partial_t \big(h_i^\varepsilon v_i^\varepsilon \big) + \nabla_x \cdot \big(q_i^\varepsilon \otimes v_i^\varepsilon \big) + \dfrac{h_i^\varepsilon}{\rho_i} \nabla_x p_i^\varepsilon = 0. \end{cases} \tag{5}$$

The regularized model (5) still satisfies the conservation of the column of water (3), and instead of the conservation law (4), it satisfies the following total energy balance:

$$\partial_t E^\varepsilon + \nabla_x \cdot \sum_{i=1}^{L} \left(\left(2\mathcal{E}_i^\varepsilon + \mathcal{K}_i^\varepsilon\right) \frac{q_i^\varepsilon}{h_i^\varepsilon} \right) = \sum_{i=1}^{L} \left(q_i^\varepsilon - h_i^\varepsilon v_i^\varepsilon\right) \cdot \nabla_x p_i^\varepsilon . \tag{6}$$

Choosing the effective discharge q_i^ε such that the right-hand side is negative, Eq. (6) leads to the dissipation of the total energy, which acts for the mathematical entropy of the system. More precisely, we set $q_i^\varepsilon = h_i^\varepsilon v_i^\varepsilon - \varepsilon\gamma \frac{H^\varepsilon}{\rho_i} \nabla_x \pi_i^\varepsilon$ with the regularizing pressure π_i^ε being a function of $(h_j^\varepsilon)_{j\in[\![1,L]\!]}$, the total water elevation $H^\varepsilon = \sum_{i=1}^{L} h_i^\varepsilon$, a dimensionless parameter $\gamma > 0$ defined further and a time scale $\varepsilon > 0$. Assuming that the regularizing function π_i^ε is smooth enough, the shallow-water model (1) is formally recovered as the limit ε goes to zero. A trivial choice of the regularizing pressure is $\pi_i^\varepsilon := p_i^\varepsilon$ and leads to a notable estimation of the variation of the pressure, i.e.

$$\int_0^T \int_{\mathbb{R}^2} \varepsilon\gamma H^\varepsilon \sum_{i=1}^{L} \frac{\|\nabla_x p_i^\varepsilon\|^2}{\rho_i} \,\mathrm{d}x\,\mathrm{d}t \le \int_{\mathbb{R}^2} E(0,x)\,\mathrm{d}x.$$

We discuss further about a more effective choice in term of computational efficiency.

Let consider a tessellation of $\mathbb{R}^2$ denote $\mathbb{T}$. For any polygon $k \in \mathbb{T}$, we denote by $|k|$ its area and by $\mathbb{F}_k$ its set of edges. In addition, for any edge $f \in \mathbb{F}_k$, we denote by $|f|$ its length and by $k_f \in \mathbb{T}$ the neighbor of k such that $k \cap k_f = f$. Similarly, we introduce a time discretization with Δt the time step such that $t^{n+1} = t^n + \Delta t$. The numerical strategy presented in this work is based on a cell-centered finite volume methods, i.e. the numerical unknowns are, in each polygon $k \in \mathbb{T}$ and each layer $1 \le i \le L$, the averaged value of the layer thickness $h_{i,k}^n$ and of the horizontal velocity $v_{i,k}^n$, where the superscript n is relative to the discrete time t^n.

The mass conservation of (5) could be compared to an advection–diffusion equation, which is classically discretized using an implicit time scheme with an upwind scheme decentered with respect to the velocity $v_{i,f}^n$ for the advection part and a centered scheme for the diffusion part. Then, the momentum balanced of (5) could be estimated explicitly. The numerical scheme reads:

$$\begin{aligned}
&\frac{h_{i,k}^{n+1} - h_{i,k}^n}{\Delta t} + \frac{1}{|k|} \sum_{f\in\mathbb{F}_k} \left(\left(q_{i,f}^{n+1} \cdot n_f^k\right)^{out} - \left(q_{i,f}^{n+1} \cdot n_f^k\right)^{in} \right) |f| = 0 \\
&\frac{h_{i,k}^{n+1} v_{i,k}^{n+1} - h_{i,k}^n v_{i,k}^n}{\Delta t} + \frac{1}{|k|} \sum_{f\in\mathbb{F}_k} \left(v_{i,k}^n \left(q_{i,f}^{n+1} \cdot n_f^k\right)^{out} - v_{i,k_f}^n \left(q_{i,f}^{n+1} \cdot n_f^k\right)^{in} \right) |f| = -\frac{1}{|k|} \frac{h_{i,k}^{n+1}}{\rho_i} \sum_{f\in\mathbb{F}_k} p_{i,f}^{n+1} |f|
\end{aligned} \tag{7}$$

with the discrete effective discharge outgoing and coming in the volume k through the face f such as

$$\left(q_{i,f}^{n+1} \cdot n_f^k\right)^{out} := h_{i,k}^{n+1} \left(v_{i,f} \cdot n_f^k\right)^+ + 2\gamma_f \frac{\varepsilon}{\Delta x_f} \frac{H_f^n}{\rho_i} \left(\delta\pi_{i,f}^{n+1} \cdot n_f^k\right)^- =: \left(q_{i,f}^{n+1} \cdot n_f^{k_f}\right)^{in}$$

and the positive and negative part functions, $2\phi^\pm = |\phi| \pm \phi \ge 0$. The following notations at the faces are used: $2\phi_f = \phi_k + \phi_{k_f}$ and $2\delta\phi_f = (\phi_{k_f} - \phi_k) n_f^k$ so that $\phi_k = \phi_f - \delta\phi_f \cdot n_f^k$ with n_f^k is the unit vector normal to the face f outward the control volume k. The characteristic length is defined by $|\partial k| \Delta x_k = |k|$ with $|\partial k| = \sum_{f\in\mathbb{F}_k} |f|$. In addition, the extremum values are defined by

$$\phi_{\max}^n = \max_{k\in\mathbb{T}} \max_{i\in[\![1,L]\!]} \left(\phi_{i,k}^n\right), \qquad \phi_{\min}^n = \min_{k\in\mathbb{T}} \min_{i\in[\![1,L]\!]} \left(\phi_{i,k}^n\right) \quad \text{and} \quad \delta\phi_{\max}^n = \max_{f\in\mathbb{F}} \max_{i\in[\![1,L]\!]} \left(\left\|\delta\phi_{i,f}^n\right\|\right).$$

From now on, we set the time scale $\varepsilon := \Delta t$.

The main result of this note is the following stability result:

Proposition 1. *Assume that there exists a density $\overline{\rho} > 0$ such that the regularizing pressure $\pi_{i,k}^{n+1}$ satisfy*

$$\sum_{i=1}^{L} \frac{\delta\pi_{i,f}^{n+1} \cdot \delta p_{i,f}^{n+1}}{\rho_i} \ge \overline{\rho} \left(g \left\|\delta \mathbf{h}_f^{n+1}\right\|\right)^2 \quad \textit{with } \delta\mathbf{h}_f^{n+1} = \left(\frac{h_{1,k_f}^{n+1} - h_{1,k}^{n+1}}{2}, \dots, \frac{h_{L,k_f}^{n+1} - h_{L,k}^{n+1}}{2} \right)^\top . \tag{8}$$

Then, defining the diffusion parameter by

$$\gamma_f = \frac{1}{2} \left(\frac{\widetilde{H}_f^{n+1}}{H_f^n} + \frac{\widetilde{V}_f^n \Delta x_f}{g H_f^n \Delta t} \right) \quad \textit{with } \widetilde{H}_f^{n+1} = \frac{\Delta x_f}{2} \sum_{i=1}^{L} \left(\left(\frac{h_{i,k}^{n+1}}{\Delta x_k} + \frac{h_{i,k_f}^{n+1}}{\Delta x_{k_f}} \right) \sum_{j=1}^{L} \frac{\rho_{\min(i,j)}}{\overline{\rho}} \right) \textit{ and}$$

$$\widetilde{V}_f^n = \sum_{i=1}^{L} \frac{\rho_{\min(i,j)}}{\overline{\rho}} \max\left(\left|v_{i,k}^n \cdot n_f^k\right|, \left|v_{i,k_f}^n \cdot n_f^k\right| \right) \tag{9}$$

and under the following CFL condition $(v^n_{\max}+\alpha\sqrt{\frac{\delta\pi^{n+1}_{\max}}{\rho_1}})\frac{\Delta t}{\Delta x_{\min}}\le\beta$ *with*

$$\alpha=\frac{L}{2}\sqrt{\frac{\rho_L}{\overline{\rho}}\left(1+\frac{\Delta x_{\max}}{\Delta x_{\min}}\right)}\quad\text{and}\quad\beta=\frac{h^{n+1}_{\min}}{2(h^{n+1}_{\max}+L\frac{\rho_L}{\rho_1}\frac{\delta\pi^{n+1}_{\max}}{g\overline{\rho}})},$$

the numerical strategy keeps the layer thickness non negative, satisfy the steady state of the lake at rest $(h_i,u_i)=(Cst,0)$ *and satisfies the dissipation of the discrete total energy* $E^n_k=\sum_{i=1}^L(\mathcal{E}^n_{i,k}+\mathcal{K}^n_{i,k})$

$$E^{n+1}_k-E^n_k+\frac{\Delta t}{|k|}\sum_{f\in\mathbb{F}_k}\sum_{i=1}^{L}\left(G^{n+1}_{\mathcal{K},i,f}\cdot n^k_f+G^{n+1}_{\mathcal{E},i,f}\cdot n^k_f+\frac{\Delta t}{2\rho_i}\left(\frac{h^{n+1}_{i,k_f}}{\Delta x_{k_f}}-\frac{h^{n+1}_{i,k}}{\Delta x_k}\right)\|\delta p^{n+1}_{i,f}\|^2\right)|f|\le 0$$

with the discrete potential energy being $\mathcal{E}^n_{i,k}=\frac{1}{2}h^n_{i,k}p^n_{i,k}$, *the discrete kinetic energy being* $\mathcal{K}^n_{i,k}=\frac{1}{2}\rho_i h^n_{i,k}\|v^n_{i,k}\|^2$, *the discrete flux of potential energy* $G^{n+1}_{\mathcal{E},i,f}$ *and the discrete flux of kinetic energy* $G^{n+1}_{\mathcal{K},i,f}$ *being respectively*

$$G^{n+1}_{\mathcal{E},i,f}=p^{n+1}_{i,f}q^{n+1}_{i,f}\cdot n^k_f-\delta h^{n+1}_{i,f}\cdot\delta p^{n+1}_{i,f}v^{n+1}_{i,f}-h^{n+1}_{i,f}\delta p^{n+1}_{i,f}\cdot n^k_f\frac{v^n_{i,k_f}-v^n_{i,k}}{2}\quad\text{and}$$
$$G^{n+1}_{\mathcal{K},i,f}=\frac{1}{2}\rho_i\|v^n_{i,k}\|^2(q^{n+1}_{i,f}\cdot n^k_f)^{out}-\frac{1}{2}\rho_i\|v^n_{i,k_f}\|^2(q^{n+1}_{i,f}\cdot n^k_f)^{in}.$$

In addition, the discrete water volume of each layer and the discrete momentum of the column of water are preserved (discrete version of (3)).

Note that the CFL condition of Proposition 1 is not optimal. However, in ocean environment, the water level is enough regular to make the celerity of the variation of pressure $\sqrt{\frac{g\delta\pi_{\max}}{\overline{\rho}}}$ small. In addition, the velocity $v_{\max}$ is small and make together the CFL condition of Proposition 1 not restrictive, contrary to the CFL of a Riemann scheme limited by the gravity wave $\sqrt{gh_{\max}}$, large in this context.

The current version of the numerical scheme satisfies the steady state of the lake at rest without topography or other source terms. In a further work [12], we present an extension of the scheme satisfying the conservation of the general steady state at rest and an adaptation to the geostrophic quasi-steady state.

On a practical point of view, the layer thickness equation is non-linear through the diffusion parameter γ_f. We use a quasi-Newton fixed point estimating the diffusion parameter γ_f with the layer thickness of the previous fixed point iteration. The resulting linear system corresponds to a M-matrix (linear advection–diffusion operator) easily solvable using classical numerical tools. However, for oceanic simulations, the number of layers L could be large and the system becomes costly to solve since the layer thickness equations are a priori coupled through the regularizing function π^ε_i. The computational efficiency of the strategy could be notably improved using a scheme estimating independently layer thickness. The simple choice $\pi^\varepsilon_i:=g\rho_i h^\varepsilon_i$ satisfies hypothesis (8) of Proposition 1. In this way, the numerical scheme presents the advantage of the uncoupled Riemann strategy [3], while preserving the conservation law of the discrete momentum of the column of water and entropy dissipation. Eventually, the numerical strategy presented does not require any hypothesis on the flow regime as long as the flow is well stratified, i.e. $\rho_1<\cdots<\rho_L$. More precisely, it could be used as well for strongly stratified densities with larger Froude numbers.

3. Numerical results

In this section, we compare the numerical solution obtained using (7) to the linear solution in a low-Froude number regime in a one-dimensional framework. The following numerical results are estimated using a coarse grid with 10 points by period, i.e. $|k|=\Delta x=10^{-1}$. We consider the following initial condition in the whole 1D space:

$$h_1=500-\cos(2\pi x),\qquad h_2=500,\qquad v_1=v_2=0,\qquad\frac{\rho_2}{\rho_1}=\frac{1}{2}.$$

In Fig. 1, we plot the interface elevations at $x=0$ as a function of time. The reference solution (red line "Asymptotic solution") is obtained by assuming that the solution is regular enough and neglecting terms of the order of the square of the Froude number. The numerical solution "Rusanov" (blue line with squares) are obtained using a Rusanov solver. For the best CFL condition (line with empty squares called "Rusanov CFL = 1"), the Rusanov scheme is not able to recover the large frequencies. The second simulation is obtained with a time step ten times smaller (line with filled squares called "Rusanov CFL = 0.1") and leads to more dissipative results. Using a Riemann solver, the smaller the time step is, the more dissipative the resolution is. The numerical strategy (7) used with the diffusion parameter γ_f given by (9) is denoted "LowFroude" in Fig. 1 (black line). The first simulation is obtained using the larger time step satisfying the CFL condition of Proposition 1 (black line without triangles called "LowFroude Stab"). The time step is too large to approach the gravity waves and the

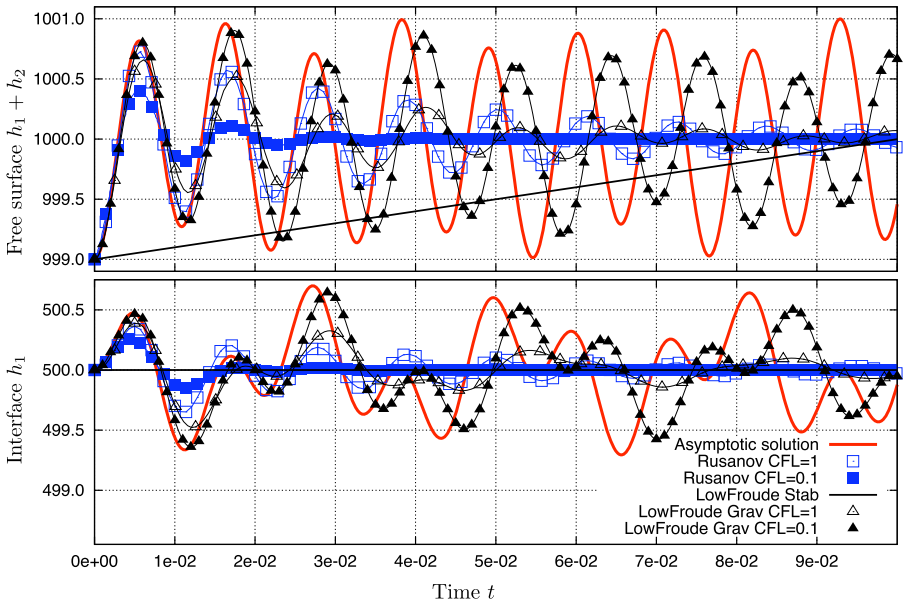

Fig. 1. (Color online.) Interface elevations in function of time.

solution only recovers the main term of the solution, i.e. the averaged value of the layer thickness. The second simulation (black line with empty triangles called "LowFroude Grav CFL = 1") is obtained setting the time step to the same order as the time step with a Riemann solver scheme. More precisely, we set $\Delta t = \frac{\Delta x}{\sqrt{gH^\star}}$ with the characteristic layer thickness $H^\star = 1000$. The numerical energy dissipation is similar to the case with the Riemann solver "Rusanov CFL = 1"; however, the large frequencies are recovered. Finally, the last simulation is obtained using a fine time discretization, i.e. $\Delta t = 10^{-1}\frac{\Delta x}{\sqrt{gH^\star}}$ (black line with filled triangles "LowFroude Grav CFL = 0.1") and leads to a better conservation of the system energy. Using the numerical scheme (7), the smaller the time step is, the less dissipative the resolution is.

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