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Functional analysis

Positivity improvement and Gaussian kernels

Positivité améliorée et noyaux gaussiens

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A B S T R A C T

We show that a positivity improving property of multilinear operators with Gaussian kernels can be determined, with sharp constants, by testing Gaussian functions only. This result can be considered as a reversed form of Lieb's theorem on maximizers of Gaussian kernels.

R É S U M É

Nous montrons qu'une propriété d'amélioration de la positivité par les opérateurs multilinéaires à noyaux gaussiens peut être déterminée, avec des constantes exactes, en testant l'opérateur uniquement sur les fonctions gaussiennes. Ce résultat peut être considéré comme une forme inverse du théorème de Lieb sur les maximiseurs des noyaux gaussiens.

1. Introduction and main result

The existence, uniqueness and structure of maximizers of integral operators with Gaussian kernels was studied in details by Lieb in [16]. His results allowed one to recover several important inequalities as the sharp Hausdorff–Young inequality [8], or Gaussian hypercontractivity [18]. They also provided the first proof of the multidimensional Brascamp–Lieb inequality [11], a multilinear inequality that unifies many classical facts in Analysis (e.g., the Loomis–Whitney inequality [17], the sharp Young convolution inequality [8,11]) and is a rich source of applications in convex geometry [2–4]. See e.g. [13,9,12,7,15,14] for recent developments. One of Lieb's theorems, the closest in spirit to the findings of this note, ensures that if $p_1, \dots, p_m \in (1, \infty)$, then the norm of a multilinear integral functional with a real Gaussian kernel, acting on functions from L^{p_i} spaces, can be determined by testing the functional on centered Gaussian functions. More specifically, let $m \geq 1$ be an integer and $H, H_1, \dots, H_m$ be Euclidean spaces endowed with the usual Lebesgue measure. For $i = 1, \dots, m$

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set $c_i = 1/p_i \in (0,1)$ and let $B_i : H \to H_i$ be a surjective linear map. Further, let $Q : H \to H$ be self-adjoint and positive semidefinite. For integrable functions $f_i : H_i \to [0,\infty)$ with $\int_{H_i} f_i > 0$ define

$$J(f_1,\dots,f_m) = \frac{\int_H \mathrm{e}^{-\pi\langle x, Qx\rangle} \prod_{i=1}^m f_i^{c_i}(B_i x)\,\mathrm{d}x}{\prod_{i=1}^m (\int_{H_i} f_i)^{c_i}}. \tag{1}$$

Then the result of Lieb asserts that

$$\sup_{f_1,\dots,f_m} J(f_1,\dots,f_m) = \sup_{g_1,\dots,g_m} J(g_1,\dots,g_m), \tag{2}$$

where g_i is a centered Gaussian function, i.e. a function of the form $\mathrm{e}^{-\langle x, A_i x\rangle}$ where $A_i : H_i \to H_i$ is positive definite.

Our main result can be considered as a reversed version of Lieb's principle (2). Namely, throughout the rest of this note, let $0 \le m^+ < m$ be integers, $c_1,\dots,c_{m^+} > 0$, $c_{m^++1},\dots,c_m < 0$, $B_i : H \to H_i$ be surjective linear maps as above, and $Q : H \to H$ be *any* self-adjoint operator with $s^+(Q)$ positive eigenvalues (counting with multiplicities). Consider the functional J defined by (1) acting on *positive* integrable functions $f_1,\dots,f_m$. In the presence of negative exponents c_i (for $i > m^+$), we will be concerned with *minimizing* the functional J, as the classical examples of the reversed Hölder and the reversed Young inequality suggest (see e.g. [11,6] and the example below).

Theorem 1.1. *If $c_1,\dots,c_{m^+} > 0$, $c_{m^++1},\dots,c_m < 0$ with $0 \le m^+ < m$ and*

$$s^+(Q) + \dim H_1 + \dots + \dim H_{m^+} \le \dim H \tag{3}$$

then with notation of (1),

$$\inf_{f_1,\dots,f_m > 0} J(f_1,\dots,f_m) = \inf_{g_1,\dots,g_m} J(g_1,\dots,g_m), \tag{4}$$

where g_i is a centered Gaussian function. Moreover, if (3) does not hold then either $J(f_1,\dots,f_m) = \infty$ for any positive integrable $f_1,\dots,f_m$ or $\inf_{f_1,\dots,f_m>0} J(f_1,\dots,f_m) = 0$.

Remark. Recently, Chen, Dafnis and Paouris [14] obtained a similar result using semigroup methods and Gaussian analysis. While especially the Gaussian formulation of their inequality (see [14, Theorem 1]) is very appealing, it is less general than our result. Indeed, it covers only a specific family of Gaussian kernels $\mathrm{e}^{-\pi\langle x, Qx\rangle}$, for which a *geometric condition* (discussed in Section 2) or its linearly equivalent form (see e.g. [9, Section 3]) is satisfied.

Examples. The Borell's reverse Gaussian hypercontractivity [10] states that for any $p, q \in (-\infty, 1)$ the operators of the Ornstein–Uhlenbeck semigroup $P_t f(x) = \int f(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}y)\gamma(\mathrm{d}y)$, where γ is a standard Gaussian measure, satisfy

$$\|P_t f\|_{L^q(\gamma)} \ge \|f\|_{L^p(\gamma)}$$

for all *positive* functions $f \in L^1(\gamma)$ if and only if $\mathrm{e}^{-2t} \le \frac{1-p}{1-q}$. For negative indices p, q, the above inequality quantifies a property of positivity improvement along the semigroup. Excluding the case when either p, q or t is 0 and using the fact that for $q \in (-\infty, 1)$ and $h \in L^q$, $\|h\|_{L^q} = \inf\{\int hk : k > 0, \int k^{q'} = 1\}$ where $q' = q/(q-1)$, the above estimate can be restated in terms of a lower bound on a bilinear integral functional: with $x = (x_1, x_2) \in \mathbb{R}^2$, $c_1 = 1/p, c_2 = 1/q' \in \mathbb{R} \setminus [0,1]$ and $Q = \frac{1}{1-\mathrm{e}^{-2t}}\begin{pmatrix} 1-(1-\mathrm{e}^{-2t})c_1 & -\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & 1-(1-\mathrm{e}^{-2t})c_2 \end{pmatrix}$,

$$\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{1}{2}\langle x, Qx\rangle} f^{c_1}(x_1) g^{c_2}(x_2)\,\mathrm{d}x \ge (2\pi)^{1-\frac{c_1+c_2}{2}} \sqrt{1-e^{-2t}} \left(\int f\right)^{c_1} \left(\int g\right)^{c_2} \tag{5}$$

holds for any positive integrable functions f, g if and only if $c_1 c_2 \det Q \ge 0$. It is easy to check that for any $c_1, c_2 \in \mathbb{R} \setminus [0,1]$, $c_1 c_2 \det Q \ge 0$ implies the hypothesis (3) and thus in the light of Theorem 1.1, (5) can be verified by testing on centered Gaussian functions f and g.

Another instance of Theorem 1.1 is the sharp reversed Young inequality [11,6], which describes the best $C_{p,q}$ such that for any $p, q, r \in (0,1]$ satisfying $1/p + 1/q = 1 + 1/r$ and any *positive* functions $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$,

$$\|f * g\|_{L^r(\mathbb{R})} \ge C_{p,q} \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}. \tag{6}$$

Also, using the same method as the one used in [11] to derive the Prékopa–Leindler inequality from (6), the reverse Brascamp–Lieb inequality of [5] can be recovered as a limit case of Theorem 1.1.

2. Optimal constants and the geometric condition

For given Gaussian functions $g_{A_i}(x) = \mathrm{e}^{-\pi\langle x, A_i x\rangle}$, where $A_i: H_i \to H_i$ is positive definite, it is easy to compute $J(g_{A_1}, \dots, g_{A_m})$. Namely, let

$$\Lambda = \left\{(A_1, \dots, A_m): A_i: H_i \to H_i \text{ and } Q + \sum_{i\le m} c_i B_i^* A_i B_i: H \to H \text{ are positive definite}\right\}.$$

If $(A_1, \dots, A_m) \in \Lambda$ then

$$J(g_{A_1}, \dots, g_{A_m}) = \left(\frac{\det(Q + \sum_{i\le m} c_i B_i^* A_i B_i)}{\prod_{i\le m}(\det A_i)^{c_i}}\right)^{-1/2} \tag{7}$$

and if $(A_1, \dots, A_m) \notin \Lambda$ then clearly $J(g_{A_1}, \dots, g_{A_m}) = \infty$. Therefore the infimum of J over centered Gaussian functions as on the right-hand side of (4) equals $D^{-1/2}$, where

$$D = \sup\left\{\frac{\det(Q + \sum_{i\le m} c_i B_i^* A_i B_i)}{\prod_{i\le m}(\det A_i)^{c_i}}: (A_1, \dots, A_m) \in \Lambda\right\} \tag{8}$$

and $D = 0$ (and thus $D^{-1/2} = \infty$) if $\Lambda = \emptyset$. As for classical Brascamp–Lieb inequalities, the above constant can be effectively computed when the following *geometric condition* holds:

$$\forall i \le m \quad B_i B_i^* = \mathrm{id}_{H_i}, \qquad Q + \sum_{i\le m} c_i B_i^* B_i = \mathrm{id}_H. \tag{9}$$

Proposition 2.1. *If (3) and (9) hold true then the supremum in (8) is attained for $(A_1, \dots, A_m) = (\mathrm{id}_{H_1}, \dots, \mathrm{id}_{H_m})$ and thus $D = 1$.*

3. Ideas of the proofs

For the proof of Theorem 1.1, we follow the monotone transportation method from [5]. However, while implementing a Gaussian kernel Q (which was not considered in [5]) is rather straightforward, the fact that some of the exponents c_i are negative is the reason why a part of the argument is substantially different from the original case where all the exponents are positive.

First of all, observe that we can assume:

$$\{x \in H: \langle x, Qx\rangle \le 0\} \cap \bigcap_{i\le m^+} \ker B_i = \{0\}, \tag{10}$$

otherwise $J \equiv \infty$. Next, if (10) holds while (3) does not, then $\inf J = 0$, as can be seen by considering Gaussian functions (by (10), $\Lambda \ne \emptyset$; then take any $(A_1, \dots, A_m) \in \Lambda$ and the corresponding centered Gaussian functions g_{A_i}; if (3) fails then one can find $i \le m^+$ such that translating the center of mass of g_{A_i} to infinity forces $J \to 0$). Therefore in the sequel we shall assume both (3) and (10).

Next, introduce a functional I that can be regarded as dual to J. Namely, let $H_+ \subseteq H$ (and $H_- \subseteq H$) be a linear span of eigenspaces corresponding to positive (resp. negative) eigenvalues of Q, and $B_+: H \to H_+$ (resp. $B_-: H \to H_-$) be such that B_+^* (resp. B_-^*) is a canonical embedding of H_+ (resp. H_-) into H, and $Q_+: H_+ \to H_+$ and $Q_-: H_- \to H_-$ are uniquely determined by

$$Q = B_+^* Q_+ B_+ - B_-^* Q_- B_-.$$

Note that Q_+ and Q_- are positive definite. Finally, for measurable functions $h_i: H_i \to [0, \infty]$ $(i = 1, \dots, m)$ satisfying $0 < \int_{H_i} h_i < \infty$ define

$$I(h_1, \dots, h_m) = \frac{\int_H^* \inf\{\mathrm{e}^{-\pi(\langle y_+, Q_+^{-1} y_+\rangle - \langle y_-, Q_-^{-1} y_-\rangle)} \prod_{i\le m} h_i^{c_i}(y_i): B_+^* y_+ - B_-^* y_- + \sum_{i\le m} c_i B_i^* y_i = y\}\,\mathrm{d}y}{\prod_{i\le m}(\int_{H_i} h_i)^{c_i}}.$$

Since the functions h_i are allowed to attain values in $[0, \infty]$, we assume the convention that whenever $0 \cdot \infty$ occurs in the product $\prod_{i\le m} h_i^{c_i}(y_i)$, it is understood as ∞. It will also be convenient to extend the definition of the functional J to measurable functions f_i with values in $[0, \infty]$ and satisfying $0 < \int_{H_i} f_i < \infty$ with the convention that if $0 \cdot \infty$ occurs in the product $\prod_{i\le m} f_i^{c_i}(B_i x)$ then it is 0.

The value of I on centered Gaussian functions can be computed using the following lemma, which is an analog of [5, Lemma 2] in the presence of negative exponents c_i.

Lemma 3.1. *For each index i let $c_i \in \mathbb{R} \setminus \{0\}$ and $B_i : H \to H_i$ be a surjective linear map. Assume the map $(B_i)_{i:c_i>0} : H \to \bigoplus_{i:c_i>0} H_i$ is a linear isomorphism. Fix any $(A_i)_i$, where $A_i : H_i \to H_i$ are positive definite and let $A = \sum c_i B_i^* A_i B_i$. If A is positive definite then*

$$\forall y \in H, \quad \langle y, A^{-1} y \rangle = \sup\Big\{ \sum c_i \langle y_i, A_i^{-1} y_i \rangle : \sum c_i B_i^* y_i = y \Big\}.$$

Note that (3) and (10) imply that the map $(B_+, B_1, \dots, B_{m^+}) : H \to H_+ \oplus H_1 \oplus \dots \oplus H_{m^+}$ is a linear isomorphism; thus we can use Lemma 3.1 to obtain that for $(A_1, \dots, A_m) \in \Lambda$,

$$I(g_{A_1^{-1}}, \dots, g_{A_m^{-1}}) = \left(\frac{\det(Q + \sum_{i \le m} c_i B_i^* A_i B_i)}{\prod_{i \le m} (\det A_i)^{c_i}} \right)^{1/2}. \tag{11}$$

The main result easily follows from the following theorem.

Theorem 3.2. *If (3) and (10) hold and $D < \infty$, then for any measurable functions $f_i, h_i : H_i \to [0, \infty]$ satisfying $\int_{H_i} f_i = \int_{H_i} h_i = 1$ $(i = 1, \dots, m)$,*

$$J(f_1, \dots, f_m) \ge D^{-1} I(h_1, \dots, h_m). \tag{12}$$

Using (8), (7), (11) and (12), under the hypothesis of Theorem 3.2 we obtain:

$$\frac{1}{\sqrt{D}} = \inf_{(A_1,\dots,A_m) \in \Lambda} J(g_{A_1}, \dots, g_{A_m}) \ge \inf J \ge D^{-1} \sup I \ge D^{-1} \sup_{(A_1,\dots,A_m) \in \Lambda} I(g_{A_1^{-1}}, \dots, g_{A_m^{-1}}) = \frac{1}{\sqrt{D}}.$$

Sketch of the proof of Theorem 3.2. An approximation argument allows us to restrict the study to the following classes of functions: f_i for $i \le m^+$ and h_i for $i > m^+$ are positive and Lipschitz in a centered closed ball $\bar{B}_{H_i}(0, R_i)$ and zero outside the ball, whereas f_i for $i > m^+$ and h_i for $i \le m^+$ are positive and Lipschitz in the whole H_i. Next, due to Caffarelli's regularity theory of Brenier's maps (see, e.g., [1]), there exists a strictly convex function φ_i, of the class $\mathcal{C}^2(B_{H_i}(0, R_i))$ for $i \le m^+$ and $\mathcal{C}^2(H_i)$ for $i > m^+$, such that the map $x \mapsto T_i(x) := \nabla\varphi_i(x)$ pushes forward the measure $f_i(x)\mathrm{d}x$ onto the measure $h_i(y)\mathrm{d}y$.

In case of the functions φ_i for $i \le m^+$, we shall work with their convex, lower semi-continuous extensions $\bar{\varphi}_i$ to the whole H_i as ∞ outside $\bar{B}_{H_i}(0, R_i)$ and $\liminf_{y \to x} \varphi_i(y)$ for $x \in \operatorname{bd} B_{H_i}(0, R_i)$. Since $\nabla\varphi_i : B_{H_i}(0, R_i) \to H_i$ is surjective for $i \le m^+$, by an elementary convexity reasoning we obtain that a subdifferential $\partial\bar{\varphi}_i(x) = \emptyset$ for any $x \notin B_{H_i}(0, R_i)$. For the functions φ_i for $i > m^+$ note that they are Lipschitz (their gradient is bounded).

In the presence of a non-trivial kernel Q, we shall also consider two convex functions, $\varphi_+(x) = \frac{1}{2}\langle x, Q_+ x \rangle$ and $\varphi_-(x) = \frac{1}{2}\langle x, Q_- x \rangle$. Note that $T_+ := \nabla\varphi_+ : H_+ \to H_+$ is a linear map, which pushes forward the Gaussian measure $(\det Q_+)^{1/2} \mathrm{e}^{-\pi\langle x, Q_+ x \rangle}\mathrm{d}x$ onto $(\det Q_+)^{-1/2} \mathrm{e}^{-\pi\langle y, Q_+^{-1} y \rangle}dy$ (and similarly $T_- := \nabla\varphi_-$).

Next, consider the convex functions $\Phi_+ : H \to \mathbb{R} \cup \{\infty\}$ and $\Phi_- : H \to \mathbb{R}$ defined by

$$\Phi_+(x) = \varphi_+(B_+ x) + \sum_{i \le m^+} c_i \bar{\varphi}_i(B_i x), \qquad \Phi_-(x) = \varphi_-(B_- x) + \sum_{i > m^+} (-c_i)\varphi_i(B_i x),$$

and the function $\Phi(x) = \Phi_+(x) - \Phi_-(x)$. Note that Φ_+ is convex, lower semi-continuous and restricted to a convex, open set $S := \bigcap_{i \le m^+} B_i^{-1}(B_{H_i}(0, R_i))$ is $\mathcal{C}^2$. Moreover, a subdifferential $\partial\Phi_+(x) = \emptyset$ for any $x \notin S$. Further consider the map $\theta : S \to H$,

$$\theta(x) := \nabla\Phi(x) = Qx + \sum_{i \le m} c_i B_i^* T_i(B_i x).$$

Let S_+ be a subset of S on which the differential

$$\mathrm{d}\theta(x) = \nabla^2\Phi(x) = Q + \sum_{i \le m} c_i B_i^* dT_i(B_i x) B_i$$

is positive semidefinite. Note that (8) can be used to bound from above $\det \mathrm{d}\theta(x)$ for $x \in S_+$. Following the approach of [5], we perform the change of variable $y = \theta(x)$ in the integral defining $I(h_1, \dots, h_m)$. Since we aim at an upper bound on $I(h_1, \dots, h_m)$, the only ingredient needed to make the argument work is the surjectivity of $\theta|_{S_+} : S_+ \to H$. To this end, take any $y_0 \in H$ and apply the following elementary lemma to the functions Φ_+ and $\Phi_- + \langle y_0, \cdot \rangle$:

Lemma 3.3. *Let $f : H \to \mathbb{R} \cup \{\infty\}$ be convex, lower semi-continuous and $g : H \to \mathbb{R}$ be convex. If the subdifferential $\partial f(x) = \emptyset$ at every $x \notin \operatorname{int}\{f < \infty\}$ and $f(x) - g(x) \to \infty$ as $|x| \to \infty$, then $f - g$ attains its infimum at an interior point of $\{f < \infty\}$.*

The hypothesis (10) and the fact that φ_i are Lipschitz for $i > m^+$ ($\nabla\varphi_i \in B_{H_i}(0, R_i)$) imply that $\Phi(x) - \langle y_0, x\rangle \to \infty$ as $|x| \to \infty$. Therefore the above lemma yields that $\Phi(x) - \langle y_0, x\rangle$ attains its infimum at a point $x_0 \in \operatorname{int}\{\Phi_+ < \infty\} = S$. Since $\Phi - \langle y_0, \cdot\rangle$ is $\mathcal{C}^2$ in S, it means that $\nabla\Phi(x_0) = y_0$ and $\nabla^2\Phi(x_0)$ is positive semidefinite, hence that $x_0 \in S_+$. □

Sketch of the proof of Proposition 2.1. The geometric condition (9) implies that the right-hand side of (8), considered as a function on Λ, has its extremum for $A_i = \mathrm{id}_{H_i}$. To show that it has actually the maximum there, we establish its concavity property.

Lemma 3.4. *Let $N \ge n = \dim H$ and $u_1, \dots, u_N \in H$. If $u_1, \dots, u_n$ form a basis in H then*

$$\Omega = \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N : \sum_{i=1}^{n} e^{x_i} u_i \otimes u_i - \sum_{i=n+1}^{N} e^{x_i} u_i \otimes u_i \text{ is positive definite} \right\}$$

is an open convex subset of $\mathbb{R}^N$ and the function $\psi : \Omega \to \mathbb{R}$,

$$\psi(x_1, \dots, x_N) = \log\det\left(\sum_{i=1}^{n} e^{x_i} u_i \otimes u_i - \sum_{i=n+1}^{N} e^{x_i} u_i \otimes u_i \right)$$

is concave. □

This lemma has a counterpart in the case of all positive exponents, namely $\phi(x) = \log\det(\sum e^{x_i} u_i \otimes u_i)$ is convex on $\mathbb{R}^N$. While the convexity of ϕ is a consequence of the Cauchy–Binet formula and of the Cauchy–Schwarz inequality (see [5]), in our case tools from matrix analysis are needed.

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