C. R. Acad. Sci. Paris, Ser. I 352 (2014) 835–840

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebraic geometry/Analytic geometry

Curvature properties for moduli of canonically polarized manifolds—An analogy to moduli of Calabi–Yau manifolds

Propriétés de courbure des modules des variétés canoniquement polarisées—une analogie avec les modules des variétés de Calabi–Yau

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ARTICLE INFO

Article history:
Received 7 July 2014
Accepted 13 August 2014
Available online 18 September 2014

Presented by Jean-Pierre Demailly

ABSTRACT

In this note we explain an analogy of moduli of canonically polarized varieties and of Calabi–Yau manifolds, when these are equipped with Kähler–Einstein forms. Given a holomorphic family $f: \mathcal{X} \to S$ of canonically polarized varieties, the direct image sheaves $R^{n-q} f_* \Omega^p_{\mathcal{X}/S}(\mathcal{K}_{\mathcal{X}/S})$ carry induced Hermitian metrics, whose curvatures enjoy similar properties. Due to the absence of a Torelli theorem, we construct a Finsler metric in the orbifold sense in order to conclude about the hyperbolicity of the moduli stack.

RÉSUMÉ

Dans cette note, nous expliquons une analogie entre les espaces de modules des variétés canoniquement polarisées et ceux des variétés de Calabi–Yau, lorsque celles-ci sont équipées de métriques de Kähler–Einstein. Étant donné une famille $f: \mathcal{X} \to S$ de variétés canoniquement polarisées, les faisceaux images directes $R^{n-q} f_* \Omega^p_{\mathcal{X}/S}(\mathcal{K}_{\mathcal{X}/S})$ possèdent des métriques hermitiennes induites, dont les tenseurs de courbure jouissent de propriétés analogues. En raison de l'absence de théorème de type Torelli, nous construisons une métrique de Finsler au sens *orbifold* afin de pouvoir conclure à l'hyperbolicité du champ de modules.

1. Introduction

Moduli spaces of smooth, polarized Kähler varieties share various properties with bounded symmetric domains and their quotients. We will see that the invariant metric on a bounded symmetric domain (that descends to any such quotients as an orbifold metric) can be seen as a model of a distinguished metric on a moduli space. It actually occurs as such a metric for compact manifolds in the Ricci flat cases of polarized complex tori, complex symplectic manifolds, and Calabi–Yau manifolds.

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http://dx.doi.org/10.1016/j.crma.2014.08.008
1631-073X/

In complex dimension one, a functorial construction was given by A. Weil in [16], who unified classical Teichmüller theory with deformation theory in the sense of Kodaira and Spencer: infinitesimal deformations were described as elements of a cohomology group, for which a distinguished metrics on the fibers (namely the hyperbolic metrics) provided a natural Hermitian inner product. This inner product had been considered before by H. Petersson in the context of automorphic forms. The Kähler property of the Weil–Petersson metric and the negativity of its Ricci curvature were first shown by Ahlfors in [1,2], and the curvature tensor was computed by Wolpert in [17]. At this point, one could see that the Weil–Petersson metric satisfies a curvature condition that is stronger than the negativity of the sectional curvature (cf. [6])—an even stronger property was later shown by Liu, Sun and Yau in [5].

A canonical Hermitian product on the tangent space of a moduli space is mostly called (generalized) Weil–Petersson metric. Such a product is given in terms of harmonic representatives of Kodaira–Spencer classes with respect to distinguished metrics on the fibers. For canonically polarized varieties as fibers, the natural choices are Kähler–Einstein metrics of constant negative curvature. The curvature of the generalized Weil–Petersson metric was computed by Siu in [12], and in [7], and efforts were made to show its negativity. In view of the result of Viehweg and Zuo [14] on the Brody hyperbolicity of this moduli stack, it became apparent that higher cohomology groups had to be included. In [11], for a family $F : \mathcal{X} \to S$ of canonically polarized manifolds, the curvature of twisted Hodge bundles $R^{n-p} f_* \Omega^p_{\mathcal{X}/S}(\mathcal{K}_{\mathcal{X}/S})$ was computed. It satisfies a lower estimate by a term that formally equals the curvature of a period domain/moduli space of polarized Ricci-flat Kähler manifolds.

The new idea was to introduce Kodaira–Spencer maps of higher degree in [10,11]. These are notably different from those maps that arise in relation to obstruction theory and Massey products. The higher-order terms were being used to offset unwanted contributions of the curvature from lower-order terms yielding a Finsler metric of negative holomorphic curvature on any relatively compact subset of the moduli space. In this note, we draw a somewhat stronger inequality from our curvature formula which yields the negativity of the above Finsler metric implying a Kobayashi hyperbolicity of the moduli stack of canonically polarized varieties.

2. Setup

We follow the setup from [9]. Notation and all results are also contained in [11,10]. We summarize these below.

We denote by $\mathcal{X} \to S$ a holomorphic family of n-dimensional canonically polarized varieties $\mathcal{X}_s = f^{-1}(s)$ for $s \in S$, equipped with Kähler–Einstein forms ω_s of constant Ricci curvature -1 according to Yau's solution to the Calabi conjecture. These induce Hermitian metrics h_s on the canonical bundles $\mathcal{K}_{\mathcal{X}_s} = \Omega^n_{\mathcal{X}_s}$ and thus a Hermitian metric h on the relative canonical bundle $\mathcal{K}_{\mathcal{X}/S}$. We denote by $\omega_{\mathcal{X}} = -\sqrt{-1}\partial\bar\partial \log h$ the curvature form.

Theorem 1. *(See [9, Theorem 1].) The form $\omega_{\mathcal{X}} = \sqrt{-1}\partial\bar\partial \log h$ is a Kähler form on $\mathcal{X}$ provided the family is effectively parameterized, and the restrictions $\omega_{\mathcal{X}}|\mathcal{X}_s$ are equal to the Kähler–Einstein forms $\omega_{\mathcal{X}_s}$ on the fibers.*

The cohomology groups $H^{n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s}))$ carry a natural Hermitian inner product, which is induced by Kähler–Einstein metrics on the fibers applied to harmonic representatives. Let $A = A^\alpha_{\bar\beta}\partial_\alpha dx^{\bar\beta}$ be a harmonic Kodaira–Spencer form for a fixed fiber. Then the cup product together with the contraction defines mappings:

$$A \cup \lrcorner : \mathcal{A}^{0,n-p}\big(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s})\big) \to \mathcal{A}^{0,n-p+1}\big(\mathcal{X}_s, \Omega^{p-1}_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s})\big) \tag{1}$$

$$\overline{A} \cup \lrcorner : \mathcal{A}^{0,n-p+1}\big(\mathcal{X}_s, \Omega^{p-1}_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s})\big) \to \mathcal{A}^{0,n-p}\big(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s})\big). \tag{2}$$

These will be applied to *harmonic* sections ψ. On the level of cohomology, for all p we have:

$$H^{n-p}\big(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s})\big) \to H^{n-p+1}\big(\mathcal{X}_s, \Omega^{p-1}_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s})\big), \qquad \psi \mapsto H(A \cup \psi) \tag{3}$$

and

$$H^{n-p+1}\big(\mathcal{X}_s, \Omega^{p-1}_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s})\big) \to H^{n-p}\big(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s})\big), \qquad \psi \mapsto H(\overline{A} \cup \psi) \tag{4}$$

where H denotes the harmonic projection. Assuming local freeness of direct image sheaves, sections of these are given by $\bar\partial$-closed forms, whose restrictions to fibers are harmonic (cf. [9, Lemma 2]). We state the formula for the curvature for diagonal terms (the general formula follows by polarization).

Theorem 2. *(See [9, Theorem IV].) The curvature tensor for $R^{n-p} f_* \Omega^p_{\mathcal{X}/S}(\mathcal{K}_{\mathcal{X}/S})$ is given by*

$$R(A, \overline{A}, \psi, \overline{\psi}) = \int_{\mathcal{X}_s} (\Box + 1)^{-1}(A \cdot \overline{A}) \cdot (\psi \cdot \overline{\psi})\, g\, dV + \int_{\mathcal{X}_s} (\Box + 1)^{-1}(A \cup \psi) \cdot (\overline{A} \cup \overline{\psi})\, g\, dV$$
$$+ \int_{\mathcal{X}_s} (\Box - 1)^{-1}(A \cup \overline{\psi}) \cdot (\overline{A} \cup \psi)\, g\, dV. \tag{5}$$

Corollary 1.

$$R(A, \overline{A}, \psi, \overline{\psi}) \geq P_n\big(\mathrm{d}(\mathcal{X}_s)\big) \cdot \|A\|^2 \cdot \|\psi\|^2 + \big\|H(A \cup \psi)\big\|^2 - \big\|H(\overline{A} \cup \psi)\big\|^2 \tag{6}$$

where $P_n(\mathrm{d}(\mathcal{X}_s))$ is a positive function that depends on the diameter of the fiber.

3. Analogies

The Hodge bundles $R^{n-p} f_* \Omega^p_{\mathcal{X}/S}$ carry connections arising from the flat connection of $R^n f_* \mathbb{C}$. The curvatures were computed by Griffiths (cf. [15]). In the above notation, the theorem reads as follows.

Theorem 3. *(See [4, (5.4)].) The curvature tensor R_h of the Hodge bundle $E^q = R^q f_* \Omega^{n-q}_{\mathcal{X}/S}$ at a point $s \in S$ is given by*

$$R_h(A, \overline{A}, \psi, \overline{\psi}) = \big\|H(A \cup \psi)\big\|^2 - \big\|H\big((A\cup)^t \psi\big)\big\|^2, \tag{7}$$

where $(A\cup)^t : H^{n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}) \to H^{n-p-1}(\mathcal{X}_s, \Omega^{p+1}_{\mathcal{X}_s})$ denotes the adjoint map with respect to the L^2 inner product.

Remark 1. Consider both cases, the Hodge bundles with fibers $H^{n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s})$ for families of polarized Ricci-flat Kähler manifolds and the bundles with fibers $H^{n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s}))$ for families of canonically polarized varieties equipped with Kähler–Einstein metrics of constant negative curvature.

Then the adjoint maps of (1) (and (3) resp.) are (2) (and (4) resp.). So these are given by the cup product with $\overline{A}$. In particular, (7) reads

$$R_h(A, \overline{A}, \psi, \overline{\psi}) = \big\|H(A \cup \psi)\big\|^2 - \big\|H(\overline{A} \cup \psi)\big\|^2,$$

and (6) yields the analogue inequality

$$R(A, \overline{A}, \psi, \overline{\psi}) \geq \big\|H(A \cup \psi)\big\|^2 - \big\|H(\overline{A} \cup \psi)\big\|^2. \tag{8}$$

We include here the proof. Let $A = A^\alpha_{\overline{\beta}} \partial_\alpha \mathrm{d}z^{\overline{\beta}}$ be a harmonic Kodaira–Spencer form on $\mathcal{X}_s$. Since the polarization is constant in the family, the class of $A \cup \omega_{\mathcal{X}_s}$ vanishes. Because of the harmonicity of A the form $A_{\overline{\beta}\overline{\delta}}\, \mathrm{d}z^{\overline{\beta}} \wedge \mathrm{d}z^{\overline{\delta}}$ is also harmonic and hence vanishes. So the corresponding tensor $A_{\overline{\beta}\overline{\delta}}$ is symmetric. Now it follows immediately from the definition of the inner product that the maps from (1) and (2) are adjoint to each other. In the next step, the morphisms from (1) and (2) are applied to *harmonic* forms $\psi \in \mathcal{A}^{0,n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s}))$ (or $\mathcal{A}^{0,n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s})$ resp.) and $\chi \in \mathcal{A}^{0,n-p+1}(\mathcal{X}_s, \Omega^{p-1}_{\mathcal{X}_s}(\mathcal{K}_{\mathcal{X}_s}))$ (or $\mathcal{A}^{0,n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s})$ resp.). For the inner products, we get that $(H(A \cup \psi), \chi) = (A \cup \psi, \chi) = (\psi, \overline{A} \cup \chi) = (\psi, H(\overline{A} \cup \chi))$. □

4. Higher-order Kodaira–Spencer maps

We recall the situation. Let $f : \mathcal{X} \to S$ be a holomorphic, effectively parameterized family of canonically polarized varieties $\mathcal{X}_s$, $s \in S$ of complex dimension n. Denote by $\rho : T_s S \to H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s})$ the Kodaira–Spencer map.

Definition 1. The p-th Kodaira–Spencer map is defined on the symmetric product of the tangent space of S, namely $\rho^p : S^p \mathcal{T}_{S,s} \to H^p(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s})$. It is induced by the classical Kodaira–Spencer map together with the natural morphism $S^p H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s}) \to H^p(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s})$.

If S is a curve, then the p-th Kodaira–Spencer map defines a Hermitian metric on $\mathcal{T}_S$. Namely, for any tangent vector u, the Kodaira–Spencer class is represented by its harmonic representative $A \in \mathcal{A}^{0,1}(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s})$, and for $p \geq 1$ the harmonic representative $A^p := H(A \wedge \ldots \wedge A) \in \mathcal{A}^{0,p}(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s})$ is taken. The spaces of harmonic p-forms with values in the p-th exterior powers of the tangent bundle of a fiber carry a natural L^2-inner product that is induced by the Kähler–Einstein metrics on a fiber.

5. Application of the curvature estimates

Our result (Theorem 2) yields the curvature of the dual bundles $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}$. The cup product with A has to be replaced with a wedge product with A:

$$A \wedge \llcorner : \mathcal{A}^{0,p}\big(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s}\big) \to \mathcal{A}^{0,p+1}\big(\mathcal{X}_s, \Lambda^{p+1} \mathcal{T}_{\mathcal{X}_s}\big) \tag{9}$$

$$\overline{A} \wedge \llcorner : \mathcal{A}^{0,p+1}\big(\mathcal{X}_s, \Lambda^{p+1} \mathcal{T}_{\mathcal{X}_s}\big) \to \mathcal{A}^{0,p}\big(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s}\big). \tag{10}$$

Again denote by H the harmonic projection. Our estimate (8) now reads:

$$R(A, \overline{A}, \nu, \overline{\nu}) \leq -\left\| H(A \wedge \overline{\nu}) \right\|^2 + \left\| H(A \wedge \nu) \right\|^2. \tag{11}$$

(Here $A \wedge \nu \in \mathcal{A}^{(0,p+1)}(\mathcal{X}_s, \Lambda^{p+1}\mathcal{T}_{\mathcal{X}_s})$ and $\overline{A} \wedge \nu \in \mathcal{A}^{(0,p-1)}(\mathcal{X}_s, \Lambda^{p-1}\mathcal{T}_{\mathcal{X}_s})$.)

Like in [10,11], we consider $A^p = H(A \wedge \ldots \wedge A)$, and set $\nu = A^p$, $\mu = A^{p-1}$.

Identifying a tangent vector $\partial/\partial s$ with a harmonic Kodaira–Spencer form A, we get:

$$-\frac{\partial^2 \log(\|A^p\|^2)}{\partial s \partial \overline{s}} \leq R\big(A, \overline{A}, A^p, \overline{A^p}\big) / \left\| A^p \right\|^2.$$

We use the semi-norms G_p such that

$$\|A\|_p = \left\| A^p \right\|^{1/p}.$$

These are continuous, where the A^p vanish, and differentiable elsewhere. The curvatures of the induced Finsler pseudo-metrics for a tangent vector $\partial/\partial s$ corresponding to the tensor A satisfy:

$$K_p = -\frac{\partial^2 \log(\|A\|_p^2)}{\partial s \partial \overline{s}} / \|A\|_p^2 \leq R\big(A, \overline{A}, A^p, \overline{A^p}\big) / p\|A\|_p^{2+2p}.$$

The estimate [9, (60), Lemma 7] can be improved using the following inequality.

Lemma 1. *(See [13, Lemma 13(ii)].) Let $\mu \in \mathcal{A}^{(0,p-1)}(\Lambda^{p-1}\mathcal{T}_{\mathcal{X}_s})$ be harmonic, A and ν as above. Then*

$$\left\| H(\overline{A} \wedge \nu) \right\|^2 \geq \left| \big(H(A \wedge \mu), \nu\big) \right|^2 / \|\mu\|^2. \tag{12}$$

Proof. The claim follows immediately from Remark 1 and from the Cauchy–Schwarz inequality. □

We now follow our earlier arguments: For $p > 1$, we set $\nu = A^p$, and $\mu = A^{p-1}$ in (12); we apply our main estimate (11) with $A^{n+1} = 0$:

$$R\big(A, \overline{A}, A^p, \overline{A^p}\big) \leq -\frac{\|A^p\|^4}{\|A^{p-1}\|^2} + \left\| A^{p+1} \right\|^2,$$

and

$$K_p = -\frac{\partial^2 \log(\|A\|_p^2)}{\partial s \partial \overline{s}} / \|A\|_p^2 \leq \frac{1}{p}\left(-\left(\frac{\|A\|_p^2}{\|A\|_{p-1}^2}\right)^{p-1} + \left(\frac{\|A\|_{p+1}^2}{\|A\|_p^2}\right)^{p+1} \right).$$

For $p = 1$, estimates are known (cf. [12,7]); we have $\|H(A \wedge \overline{A})\|^2 = \|A\|^2 / \mathrm{vol}(\mathcal{X}_s)$ so that

$$R(A, \overline{A}, A\overline{A}) \leq -c \cdot \|A\|^2 + \frac{\|A^2\|^2}{\|A\|^2}$$

with $c := 1/\mathrm{vol}(\mathcal{X}_s)$. (In fact $c := 2/\mathrm{vol}(\mathcal{X}_s)$ can be taken.) Hence

$$K_1 \leq -c + \frac{\|A\|^4}{\|A^2\|^2}$$

Now

$$K_1 \leq -c + \frac{G_2^2}{G_1^2}; \qquad K_p \leq \frac{1}{p}\left(-\frac{G_p^{p-1}}{G_{p-1}^{p-1}} + \frac{G_{p+1}^{p+1}}{G_p^{p+1}} \right) \quad \text{for } p > 1 \text{ with } G_{n+1} := 0.$$

We use a Finsler metric H, a sum of the metrics G_p with positive coefficients of the form $p\,\alpha_p$

$$H = \sum_{p=1}^{n} p\,\alpha_p\,G_p.$$

So

$$K_H \leq \frac{1}{H^2} \sum_{p=1}^{n} p\,\alpha_p\,G_p^2 K_p$$

according to [8, Lemma 3]. This inequality implies:

$$K_H \leq \frac{1}{H^2}\left(-c\cdot\alpha_1 G_1^2 - \sum_{p=2}^{n}\left(\alpha_p \frac{G_p^{p+1}}{G_{p-1}^{p-1}} - \alpha_{p-1}\frac{G_p^p}{G_{p-1}^{p-2}}\right)\right). \tag{13}$$

We arrive at the following sharper version of [9, Prop. 15].

Theorem 4. *The above functorial construction yields a Finsler metric for effectively parameterized families of canonically polarized manifolds that descends to the moduli space in the orbifold sense. For suitable coefficients $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$, the curvature of the Finsler metric is bounded from above by a negative constant that only depends upon the dimension and the volume of the fibers.*

The theorem together with Demailly's version of the Ahlfors Lemma [3, 3.2] (cf. [9, Prop. 13]) implies the Kobayashi hyperbolicity of the moduli stack: the harmonic Kodaira–Spencer tensors A_s depend in a C^∞ way upon the parameter by [9, Prop. 2], and so do exterior powers. For a family over a (smooth) curve C, the tensors $H(A_s \wedge \ldots \wedge A_s) = A_s^p$ are locally bounded everywhere and differentiable in $s \in C$ over the complement of the discrete set of points where $\dim H^p(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s})$ is not constant.

The *proof of Theorem 4* follows in an elementary way from (13) by adjusting the constants α_p, based upon the simple inequality below.

Lemma 2. *Let $p \in \mathbb{N}$. Then*

$$f(x) = x^{p+1} - x^p - x^2/2 + 1/2 \geq 0$$

for all $x \geq 0$.

Lemma 3. *Let $\alpha_p > 0$ for $p = 1, \ldots, n$. Then for all $x_p \geq 0$*

$$\begin{aligned}&\sum_{p=2}^{n}(\alpha_p x_p^{p+1} - \alpha_{p-1}x_p^p)x_{p-1}^2\cdot\ldots\cdot x_1^2 \\ &\quad\geq \frac{1}{2}\left(-\frac{\alpha_1^3}{\alpha_2^2}x_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}x_n^2\cdot\ldots\cdot x_1^2 + \sum_{p=2}^{n-1}\left(\frac{\alpha_{p-1}^{p-1}}{\alpha_p^{p-2}} - \frac{\alpha_p^{p+2}}{\alpha_{p+1}^{p+1}}\right)x_p^2\cdot\ldots\cdot x_1^2\right)\end{aligned} \tag{14}$$

holds.

Proof. Use $\frac{\alpha_{p-1}^{p+1}}{\alpha_p^p} f(\frac{\alpha_p}{\alpha_{p-1}}x_p) \geq 0$ from Lemma 2 and take a sum. □

Proof of the Theorem. We may replace the square of a sum in H^2 by the sum of the squares relaxing the estimate by a constant positive factor. Hence it is sufficient to show that

$$-c\cdot\alpha_1 G_1^2 - \sum_{p=2}^{n}\left(\alpha_p\frac{G_p^{p+1}}{G_{p-1}^{p-1}} - \alpha_{p-1}\frac{G_p^p}{G_{p-1}^{p-2}}\right) \leq -\sum_{p=1}^{n}\gamma_p G_p^2 \tag{15}$$

for some $\gamma_j > 0$. Obviously, we can chose α_1 arbitrarily and define $\alpha_1 < \alpha_2 < \alpha_3 < \ldots < \alpha_n$ inductively so that the coefficients

$$\gamma_1 = c\cdot\alpha_1 - \frac{1}{2}\frac{\alpha_1^3}{\alpha_2^2}, \qquad \gamma_p = \frac{1}{2}\left(\frac{\alpha_{p-1}^{p-1}}{\alpha_p^{p-2}} - \frac{\alpha_p^{p+2}}{\alpha_{p+1}^{p+1}}\right) \quad \text{for } p = 2, \ldots, n-1 \quad \text{and} \quad \gamma_n = \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}$$

of $x_p^2\cdot\ldots\cdot x_1^2$ in (14) are positive. Now we set $x_p = G_p/G_{p-1}$, and (15) follows. □

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