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Partial differential equations

Remarks on a lemma by Jacques-Louis Lions

Remarques sur un lemme de Jacques-Louis Lions

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ABSTRACT

Let Ω be a bounded and connected open subset of $\mathbb{R}^N$ with a Lipschitz-continuous boundary $\partial\Omega$, the set Ω being locally on one side of $\partial\Omega$. It is shown in this Note that a fundamental characterization of the space $L^2(\Omega)$ due to Jacques-Louis Lions is in effect equivalent to a variety of other properties. One of the keys for establishing these equivalences is a specific "approximation lemma", itself one of these equivalent properties.

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RÉSUMÉ

Soit Ω un ouvert borné et connexe de $\mathbb{R}^N$ de frontière $\partial\Omega$ lipschitzienne, l'ensemble Ω étant localement du même côté de $\partial\Omega$. On montre dans cette Note qu'une caractérisation fondamentale de l'espace $L^2(\Omega)$ due à Jacques-Louis Lions est en fait équivalente à un certain nombre d'autres propriétés. L'une des clés pour établir ces équivalences est un « lemme d'approximation » spécifique, qui constitue lui-même l'une de ces propriétés équivalentes.

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1. Definitions and notations

In what follows, N designates a fixed integer ≥ 2. Unless otherwise specified, Latin indices range in the set $\{1, 2, \dots, N\}$.

The notation V' designates the dual space of a topological vector space V and ${}_{V'}\langle\cdot,\cdot\rangle_V$ designates the duality between V' and V. Given a subspace W of a normed vector space V,

$$W^0 := \left\{v' \in V';\ {}_{V'}\langle v', w\rangle_V = 0 \text{ for all } w \in W\right\}$$

designates the polar set of W; if V is a Hilbert space, $W^\perp$ designates the orthogonal complement of W.

Let Ω be an open subset of $\mathbb{R}^N$ and let $x = (x_i)$ be a generic point in Ω. Partial derivative operators of the first order, in the classical sense or in the sense of distributions, are denoted $\partial_i := \partial/\partial x_i$. The space of functions that are indefinitely

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differentiable in Ω and have compact supports in Ω is denoted $\mathcal{D}(\Omega)$ and the space of distributions on Ω is denoted $\mathcal{D}'(\Omega)$. If $f \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we also use the shorter notation $f(\varphi) := {}_{\mathcal{D}'(\Omega)}\langle f, \varphi\rangle_{\mathcal{D}(\Omega)}$. The notations $H^1(\Omega)$ and $H^1_0(\Omega)$ designate the usual Sobolev spaces, and the notation $H^{-1}(\Omega)$ designates the dual space of $H^1_0(\Omega)$ endowed with the norm of $H^1(\Omega)$. Finally, we define the space

$$L^2_0(\Omega) := \left\{ f \in L^2(\Omega);\ \int_\Omega f\,\mathrm{d}x = 0 \right\}.$$

Spaces of functions and vector fields defined over Ω are respectively denoted by italic capitals and boldface Roman capitals.

The *gradient operator* $\mathbf{grad} : \mathcal{D}'(\Omega) \to \boldsymbol{\mathcal{D}}'(\Omega)$ is defined for each $f \in \mathcal{D}'(\Omega)$ by

$${}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \mathbf{grad}\, f, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} := -{}_{\mathcal{D}'(\Omega)}\langle f, \operatorname{div} \boldsymbol{\varphi}\rangle_{\mathcal{D}(\Omega)} \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega).$$

Note that, when restricted to $L^2(\Omega)$, the mapping $\mathbf{grad} : L^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ satisfies

$${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \mathbf{grad}\, f, \boldsymbol{v}\rangle_{\boldsymbol{H}^1_0(\Omega)} = -\int_\Omega f \operatorname{div} \boldsymbol{v}\,\mathrm{d}x \quad \text{for all } f \in L^2(\Omega) \text{ and all } \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega).$$

This shows that the operator $\mathbf{grad} : L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is the dual operator of $-\operatorname{div} : \boldsymbol{H}^1_0(\Omega) \to L^2_0(\Omega)$. It also easily implies that, if the open set Ω is connected, the operator $\mathbf{grad} : L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is one-to-one.

The *curl operator* $\mathbf{curl} : \boldsymbol{\mathcal{D}}'(\Omega) \to \mathcal{D}(\Omega; \mathbb{R}^{N(N-1)/2})$ is defined for each $\boldsymbol{h} = (h_i) \in \boldsymbol{\mathcal{D}}'(\Omega)$ by

$$(\mathbf{curl}\, \boldsymbol{h})_{ij} = \partial_i h_j - \partial_j h_i \quad \text{for each } i < j.$$

A *domain* Ω in $\mathbb{R}^N$ is a bounded and connected open subset Ω of $\mathbb{R}^N$ whose boundary $\partial\Omega$ is Lipschitz-continuous, the set Ω being locally on the same side of $\partial\Omega$.

Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$ and, given $r > 0$, let $B(x; r) := \{y \in \mathbb{R}^N;\ |y - x| < r\}$. An open subset of $\mathbb{R}^N$ is *starlike with respect to an open ball* $B(x; r)$ if, for each $z \in \Omega$, the convex hull of the set $\{z\} \cup B(x; r)$ is contained in the set Ω.

2. Jacques-Louis Lions' lemma

Let Ω be a domain in $\mathbb{R}^N$. The *classical J.-L. Lions lemma* asserts that $f \in H^{-1}(\Omega)$ and $\mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega)$ implies $f \in L^2(\Omega)$. Its first published proof, under the assumption that the boundary of Ω is smooth, appeared in Duvaut and Lions [7]; see also Tartar [12] for a different proof, under the same assumption. The first proof for a general domain is due to Geymonat and Suquet [8].

That the assumption $f \in H^{-1}(\Omega)$ can be replaced by the more general assumption $f \in \mathcal{D}'(\Omega)$ was established by Borchers and Sohr [4], as a consequence of a result of Bogovskii [3], who gave a constructive proof that the operator $\operatorname{div} : \boldsymbol{H}^1_0(\Omega) \to L^2_0(\Omega)$ is onto; then by Amrouche and Girault [2], as a consequence of an inequality due to Nečas [11] (also used in [8]), asserting the existence of a constant $C_0(\Omega)$ such that

$$\|f\|_{L^2(\Omega)} \le C_0(\Omega)\big(\|f\|_{H^{-1}(\Omega)} + \|\mathbf{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)}\big) \quad \text{for all } f \in L^2(\Omega).$$

Note that the results of both [2], [4] and [8] hold for a general domain Ω. We shall call *J.-L. Lions' lemma* this stronger result, which thus asserts that

$$f \in \mathcal{D}'(\Omega) \quad \text{and} \quad \mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \quad \text{implies } f \in L^2(\Omega).$$

Both the classical J.-L. Lions lemma and its more general version above are fundamental results from functional analysis, with many crucial applications to partial differential equations (see, e.g., Sections 6.14 to 6.19 in [5]).

3. Jacques-Louis Lions' lemma and its relation to other basic results

The main objective of this Note is to show, by means of a sequence of implications (Theorems 1 to 5), that the above properties, viz., both versions of J.-L. Lions' lemma, the surjectivity of div, and the inequality of Nečas, as well as other properties, are in fact *equivalent*. The key to establishing these equivalences is an *approximation lemma* (cf. Theorem 4), which constitutes in effect one of the equivalence properties and appears to be new. In so doing, we provide in addition what we believe are substantially simpler proofs of those implications than those that are already known in the literature. Detailed proofs will appear in [1].

Theorem 1. *Let Ω be a domain in $\mathbb{R}^N$. Then the* **classical J.-L. Lions lemma:**

$$f \in H^{-1}(\Omega) \quad \text{and} \quad \mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \quad \text{implies } f \in L^2(\Omega)$$

implies **J. Nečas' inequality***: there exists a constant $C_0(\Omega)$ such that*

$$\|f\|_{L^2(\Omega)} \le C_0(\Omega)\big(\|f\|_{H^{-1}(\Omega)} + \|\mathbf{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)}\big) \quad \text{for all } f \in L^2(\Omega).$$

Sketch of proof. The space

$$K(\Omega) := \big\{f \in H^{-1}(\Omega);\ \mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega)\big\},$$

equipped with the norm $f \in K(\Omega) \to (\|f\|_{H^{-1}(\Omega)} + \|\mathbf{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)})$, is a Banach space. Since the identity mapping from $L^2(\Omega)$ into $K(\Omega)$ is onto by the *classical J.-L. Lions lemma*, J. Nečas' inequality follows from Banach's open mapping theorem. □

Theorem 2. *Let Ω be a domain in $\mathbb{R}^N$. Then J. Nečas' inequality implies that the image of the space $L^2_0(\Omega)$ under the operator* **grad** *: $L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is closed in $\boldsymbol{H}^{-1}(\Omega)$.*

Sketch of proof. The compactness of the canonical injection from $L^2(\Omega)$ into $H^{-1}(\Omega)$ (for a proof, see, e.g., Theorem 6.11-3 in [5]) and *J. Nečas' inequality* together imply the existence of a constant $c(\Omega)$ such that

$$\|f\|_{L^2(\Omega)} \le c(\Omega)\|\mathbf{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)} \quad \text{for all } f \in L^2_0(\Omega),$$

which in turn implies that the image of $L^2_0(\Omega)$ under **grad** is closed in $H^{-1}(\Omega)$ (note that this proof, which is that of Theorem 6.14-1 in [5], does not rely on the well-known Peetre–Tartar lemma as is usually the case; see, e.g., Corollary 2.1 in Chapter 1 of [9]). □

Theorem 3. *Let Ω be a domain in $\mathbb{R}^N$. That the image of $L^2_0(\Omega)$ under* **grad** *is closed in $\boldsymbol{H}^{-1}(\Omega)$ is equivalent to the following* **coarse version of the de Rham theorem** (a terminology borrowed from [9])*: given a vector field $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, there exists a function $p \in L^2_0(\Omega)$ such that*

$$\mathbf{grad}\, p = \boldsymbol{h} \quad \text{in } \boldsymbol{H}^{-1}(\Omega)$$

if

$${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{h}, \boldsymbol{v}\rangle_{\boldsymbol{H}^1_0(\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \quad \text{such that} \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega.$$

If this is the case, the function $p \in L^2_0(\Omega)$ is uniquely determined.

That the image of $L^2_0(\Omega)$ under **grad** *is closed in $\boldsymbol{H}^{-1}(\Omega)$ is also equivalent to the surjectivity of the operator*

$$\operatorname{div} : \boldsymbol{H}^1_0(\Omega) \to L^2_0(\Omega),$$

a property that in turn implies that, for each $f \in L^2_0(\Omega)$, there exists a unique element $\boldsymbol{u}_f \in (\mathbf{Ker}\operatorname{div})^{\perp} \subset \boldsymbol{H}^1_0(\Omega)$ such that

$$\operatorname{div} \boldsymbol{u}_f = f,$$

and that there exists a constant $C_1(\Omega)$ such that the linear operator $f \in L^2_0(\Omega) \to \boldsymbol{u}_f \in (\mathbf{Ker}\operatorname{div})^{\perp}$ defined in this fashion satisfies

$$\|\boldsymbol{u}_f\|_{\boldsymbol{H}^1(\Omega)} \le C_1(\Omega)\|f\|_{L^2(\Omega)} \quad \text{for all } f \in L^2_0(\Omega).$$

Sketch of proof. These equivalences follow from the properties that **grad** $: L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is one-to-one and is the dual operator of $-\operatorname{div} : \boldsymbol{H}^1_0(\Omega) \to L^2_0(\Omega)$, and from Banach's closed range theorem; the existence of a constant $C_1(\Omega)$ follows from Banach's open mapping theorem. □

Theorem 4. *Let Ω be a domain in $\mathbb{R}^N$ that is starlike with respect to an open ball. That $\operatorname{div} : \boldsymbol{H}^1_0(\Omega) \to L^2_0(\Omega)$ is onto implies that the following* **approximation lemma** *holds: There exists a constant $C_2(\Omega)$ such that, given any function*

$$\varphi \in \mathcal{D}_0(\Omega) := \left\{\varphi \in \mathcal{D}(\Omega);\ \int_\Omega \varphi\, \mathrm{d}x = 0\right\} \subset L^2_0(\Omega),$$

there exist vector fields $\boldsymbol{v}_n = \boldsymbol{v}_n(\varphi) \in \boldsymbol{\mathcal{D}}(\Omega)$, $n \ge 1$, such that

$$\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \le C_2(\Omega)\|\varphi\|_{L^2(\Omega)} \quad \text{for all } n \ge 1 \quad \text{and} \quad \operatorname{div} \boldsymbol{v}_n \to \varphi \quad \text{in } \mathcal{D}(\Omega) \text{ as } n \to \infty.$$

Sketch of proof. Let $\varphi \in \mathcal{D}_0(\Omega)$. By the assumed *surjectivity of* $\operatorname{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$, there exists a unique element $\boldsymbol{u}_\varphi \in (\mathbf{Ker}\operatorname{div})^\perp \subset \boldsymbol{H}_0^1(\Omega)$ such that

$$\operatorname{div} \boldsymbol{u}_\varphi = \varphi \quad \text{and} \quad \|\boldsymbol{u}_\varphi\|_{\boldsymbol{H}^1(\Omega)} \le C_1(\Omega)\|\varphi\|_{L^2(\Omega)}.$$

Let $\boldsymbol{w}$ denote the extension of $\boldsymbol{u}_\varphi$ by $\mathbf{0}$ outside Ω, so that $\boldsymbol{w} \in H^1(\mathbb{R}^N)$, $\operatorname{div} \boldsymbol{w} = \varphi$ in Ω, $\operatorname{div} \boldsymbol{w} = 0$ outside Ω, and $\|\boldsymbol{w}\|_{H^1(\mathbb{R}^N)} = \|\boldsymbol{u}_\varphi\|_{\boldsymbol{H}^1(\Omega)}$.

Assume without loss of generality that the ball with respect to which Ω is star-shaped is centered at the origin. Let n_0 denote the smallest integer satisfying $n_0 > \frac{2}{r}$ and let $\lambda_n := 1 - \frac{2}{nr}$, $n \ge n_0$, where r is the radius of the ball, let $\boldsymbol{u}_n(x) := \lambda_n \boldsymbol{w}(\frac{x}{\lambda_n})$ for all $x \in \mathbb{R}^N$, $n \geqslant n_0$, and let $(\rho_n)_{n=1}^\infty$ denote a family of *mollifiers* $\rho_n \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} \rho_n \subset \overline{B(0; \frac{1}{n})}$. Then each *convolution product* $\boldsymbol{w}_n := \boldsymbol{u}_n \star \rho_n \in \mathcal{C}^\infty(\mathbb{R}^N)$, $n \ge n_0$, has the following properties:

$$\boldsymbol{v}_n := \boldsymbol{w}_n|_\Omega \in \boldsymbol{\mathcal{D}}(\Omega), \quad \operatorname{div} \boldsymbol{v}_n = \varphi\left(\frac{\cdot}{\lambda_n}\right) \star \rho_n \quad \text{and} \quad \|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \le \|\boldsymbol{u}_\varphi\|_{\boldsymbol{H}^1(\Omega)}.$$

Finally, one shows that $\operatorname{div} \boldsymbol{v}_n \to \varphi$ in $\mathcal{D}(\Omega)$ (equipped with its natural inductive limit topology) as $n \to \infty$ by combining differentiability properties of mollifiers with the uniform continuity and the boundedness of each higher-order partial derivative of the function φ. $\square$

Theorem 5. *Let Ω be a domain in $\mathbb{R}^N$. Then the approximation lemma implies that* **J.-L. Lions' lemma** *holds, viz.,*

$$f \in \mathcal{D}'(\Omega) \quad \textit{and} \quad \mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \textit{ implies } f \in L^2(\Omega).$$

Sketch of proof. Assume first that Ω is starlike with respect to an open ball, and let $f \in \mathcal{D}'(\Omega)$ be such that $\mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega)$. Hence there exists a constant $C_2(\Omega)$ such that

$$\left|{}_{\mathcal{D}'(\Omega)}\langle f, \operatorname{div} \boldsymbol{\psi}\rangle_{\mathcal{D}(\Omega)}\right| = \left|{}_{\boldsymbol{H}^{-1}(\Omega)}\langle \mathbf{grad}\, f, \boldsymbol{\psi}\rangle_{\boldsymbol{H}_0^1(\Omega)}\right| \le C_2(\Omega)\|\boldsymbol{\psi}\|_{\boldsymbol{H}^1(\Omega)} \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega).$$

Let $\varphi_1 \in \mathcal{D}(\Omega)$ be such that $\int_\Omega \varphi_1 \,\mathrm{d}x = 1$ and let φ be an arbitrary function in $\mathcal{D}(\Omega)$. Then

$$\varphi_0 := \left(\varphi - \left(\int_\Omega \varphi \,\mathrm{d}x\right)\varphi_1\right) \in \mathcal{D}_0(\Omega),$$

and thus by the *approximation lemma*, there exist vector fields $\boldsymbol{v}_n \in \boldsymbol{\mathcal{D}}(\Omega)$, $n \ge 1$, such that

$$\operatorname{div} \boldsymbol{v}_n \to \varphi_0 \quad \text{in } \mathcal{D}(\Omega) \text{ as } n \to \infty \quad \text{and} \quad \|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \le C_1(\Omega)\|\varphi_0\|_{L^2(\Omega)}, \quad n \ge 1.$$

Hence

$$f(\operatorname{div} \boldsymbol{v}_n) \to f(\varphi_0) \quad \text{as } n \to \infty \quad \text{and} \quad \left|{}_{\mathcal{D}'(\Omega)}\langle f, \operatorname{div} \boldsymbol{v}_n\rangle_{\mathcal{D}(\Omega)}\right| \le C_2(\Omega)\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \quad \text{for each } n \ge 1.$$

Since $\|\varphi_0\|_{L^2(\Omega)} \le (1 + \|\varphi_1\|_{L^2(\Omega)}\sqrt{\operatorname{meas}\,\Omega})\|\varphi\|_{L^2(\Omega)}$ and $f(\varphi) = f(\varphi_0) + (\int_\Omega \varphi \,\mathrm{d}x) f(\varphi_1)$, it follows that there exists a constant $C_3(\Omega)$ such that

$$\left|f(\varphi)\right| \le C_3(\Omega)\|\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in \mathcal{D}(\Omega),$$

which proves that $f \in L^2(\Omega)$. Hence *J.-L. Lions' lemma* holds for domains that are *starlike with respect to an open ball.*

If now Ω is a general domain, Ω can be written as a finite union $\bigcup_{j=1}^J \Omega_j$ of domains Ω_j, $j \in J$, contained in Ω, each one of which is starlike with respect to an open ball. Then *J.-L. Lions' lemma* on Ω is obtained by combining the above J.-L. Lions lemma on each set Ω_j, $1 \leqslant j \leqslant J$, with a partition of unity associated with the open cover $\Omega = \bigcup_{j=1}^J \Omega_j$. $\square$

Since J.-L. Lions' lemma evidently implies that the classical J.-L. Lions lemma holds, *all the properties established in* Theorems 1 *to* 5 *are thus equivalent.*

There exist *independent*, i.e., "direct", proofs of some of these properties, for instance that by Bogovskii [3] of the surjectivity of $\operatorname{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ or that by Nečas [11] of Nečas inequality. Therefore, any such proof provides, by means of some of the equivalences established here, a means of proving J.-L. Lions' lemma, the known "direct" proofs of which are notoriously difficult when Ω is a general domain.

For completeness, we also mention in Theorems 6 and 7 below two other "less direct" equivalences involving J.-L. Lions' lemma. The first one is with a "weak" version (in the sense that it holds in lower-order Sobolev spaces) of the classical Poincaré lemma.

Theorem 6. *The classical J.-L. Lions lemma and the surjectivity of* $\operatorname{div}: \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ (itself a consequence of the classical J.-L. Lions lemma; cf. Theorems 1, 2 and 3) *together imply that the following* **weak Poincaré lemma** *holds: let Ω be a simply-connected domain in $\mathbb{R}^N$. Then, given a vector field $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, there exists a function $p \in L^2(\Omega)$ such that*

$$\mathbf{grad}\, p = \boldsymbol{h} \quad \text{in } \boldsymbol{H}^{-1}(\Omega)$$

if

$$\mathbf{curl}\, \boldsymbol{h} = \mathbf{0} \quad \text{in } \boldsymbol{H}^{-2}(\Omega).$$

If this is the case, all other solutions $\tilde{p} \in L^2(\Omega)$ to $\mathbf{grad}\, \tilde{p} = \boldsymbol{h}$ *are of the form $\tilde{p} = p + C$, where C is a constant.*

Conversely, the weak Poincaré lemma implies that J.-L. Lions' lemma holds on any domain in $\mathbb{R}^N$.

Sketch of proof. For the proof of the first part of this theorem, we refer the reader either to Ciarlet and Ciarlet, Jr. [6] for the original proof, or to Kesavan [10] for a simpler proof.

To establish the second part, assume first that the domain Ω is simply-connected. Then it suffices to notice that $\mathbf{curl}\,\mathbf{grad}\, f = 0$ in $\mathcal{D}'(\Omega)$ for any $f \in \mathcal{D}'(\Omega)$, so that the assumption that $\mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega)$ allows us to use the weak Poincaré lemma. This shows that there exists a function $p \in L^2(\Omega)$ such that $\mathbf{grad}\, p = \mathbf{grad}\, f$, which in turn implies that $f - p$ is a constant function. Hence $f \in L^2(\Omega)$, i.e., J.-L. Lions' lemma holds in this case.

Since a general domain Ω in $\mathbb{R}^N$ can be written as a finite union $\Omega = \bigcup_{i \in I} \Omega_i$ of simply-connected domains $\Omega_i \subset \Omega$, it suffices to use a partition of unity associated with the open cover $\bigcup_{i \in I} \Omega_i$ of Ω and to use the above implication over each subdomain Ω_i, $i \in I$. □

The second equivalence is with a "less coarse" version of de Rham theorem than that established in Theorem 3.

Theorem 7. *J.-L. Lions' lemma and the coarse version of the de Rham theorem* (itself a consequence of J.-L. Lions' lemma; cf. Theorems 1, 2, and 3) *together imply that the following* **simplified version of the de Rham theorem holds** (a terminology again borrowed from [9])*: let Ω be a domain in $\mathbb{R}^N$. Then, given a vector field $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, there exists a function $p \in L_0^2(\Omega)$ such that*

$$\mathbf{grad}\, p = \boldsymbol{h} \quad \text{in } \boldsymbol{H}^{-1}(\Omega)$$

if

$${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}_0^1(\Omega)} = 0 \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega) \quad \text{such that} \quad \operatorname{div} \boldsymbol{\varphi} = 0 \quad \text{in } \Omega.$$

If this is the case, the function $p \in L_0^2(\Omega)$ is uniquely determined.

Conversely, the simplified version of the de Rham theorem implies that J.-L. Lions' lemma holds.

Sketch of proof. The proof of the first part uses arguments similar to those used in the proof of Theorem 2.3 in Chapter 1 of Girault and Raviart [9]. The proof of the second part follows from Theorem 3. □

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