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Dynamical systems

Partially hyperbolic diffeomorphisms on Heisenberg nilmanifolds and holonomy maps

Difféomorphismes partiellement hyperboliques de la nil-variété de Heisenberg et applications d'holonomie

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A B S T R A C T

In this note we show that all partially hyperbolic automorphisms on a 3-dimensional non-Abelian nilmanifold can be C^1-approximated by structurally stable C^∞-diffeomorphisms, whose chain recurrent set consists of one attractor and one repeller. In particular, all these partially hyperbolic automorphisms are not robustly transitive. As a corollary, the holonomy maps of the stable and unstable foliations of the approximating diffeomorphisms are twisted quasiperiodically forced circle homeomorphisms, which are transitive but non-minimal and satisfy certain fiberwise regularity properties.

R É S U M É

Dans cette note, nous démontrons que les automorphismes partiellement hyperboliques de la nil-variété non abélienne de dimension 3 peuvent tous être approchés dans la topologie C^1 par des difféomorphismes structurellement stables, chacun possédant un attracteur et un répulseur comme seuls ensembles récurrents par chaîne. Cela implique que ces automorphismes partiellement hyperboliques ne sont pas robustement transitifs. Comme corollaire, nous en déduisons que les holonomies des feuilletages stables et instables des difféomorphismes approximants sont des homéomorphismes quasi périodiquement forcés twistés du cercle, qui sont transitifs mais pas minimaux, qui satisfont à certaines propriétés de régularité dans les fibres.

1. Introduction

Let $\mathbb{H}=\{(x,y,z):\ x,y,z\in\mathbb{R}\}$ be the 3-dimensional real Heisenberg group, endowed with the group operation

$$(a,b,c)\cdot(x,y,z)=(a+x,b+y,c+z+ay).$$

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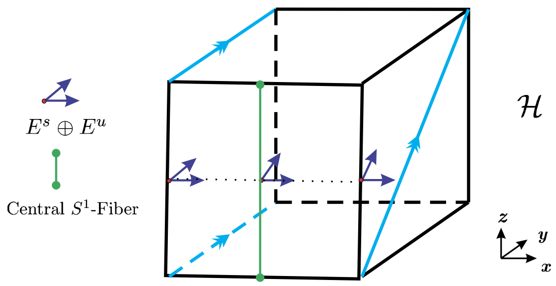

Fig. 1. (Color online.) Heisenberg nilmanifold and contact structure: the nilmanifold is obtained by identifying the left- and right-hand sides of the cube $[0,1]^3$ using a Dehn twist. The other faces are identified by standard translations, see (1).

$\mathbb{H}$ is the simplest non-Abelian Lie group. Any lattice of $\mathbb{H}$ is isomorphic to $\Gamma_k = \{(x,y,z) \in \mathbb{H}:\ x, y, \in \mathbb{Z}, z \in \frac{1}{k}\mathbb{Z}\}$, where k is a positive integer. One obtains the quotient space $\mathcal{H}_k = \mathbb{H}/\Gamma_k$ by the equivalent relation $\sim$, where $(a,b,c) \sim (x,y,z)$ if and only if $(a,b,c)\cdot(x,y,z)^{-1} \in \Gamma_k$.

In this paper, we restrict ourselves to the case when $k=1$, and denote $\Gamma = \Gamma_1$, $\mathcal{H} = \mathcal{H}_1$. Then $\mathcal{H}$ is represented as

$$\mathcal{H} = [0,1]^3/\sim = \left\{(x,y,z) \in [0,1]^3:\ (x,y,z) = (x+1,y,z+y) = (x,y+1,z) = (x,y,z+1)\right\}. \tag{1}$$

It is not hard to see that $\mathcal{H}$ is an $\mathbb{S}^1$-bundle over the torus $\mathbb{T}^2$ with Euler number 1, see Fig. 1.

Let F be a Lie group automorphism defined on $\mathbb{H}$. If F preserves the lattice Γ, then it induces a diffeomorphism f on the nilmanifold $\mathcal{H}$. To emphasize the algebraic nature, we will call f an *automorphism* on $\mathcal{H}$. An automorphism f is called *partially hyperbolic* if the tangent bundle $T\mathcal{H}$ admits a Df-invariant splitting $T\mathcal{H} = E^s \oplus E^c \oplus E^u$, and there exist an integer $k>0$ and a constant $0<\mu<1$ such that, for any $p \in \mathcal{H}$ and any unit vectors $v^s \in E^s(p)$, $v^c \in E^c(p)$, and $v^u \in E^u(p)$, we have

$$\left\|Df^k\left(v^s\right)\right\| < \mu < \left\|Df^k\left(v^c\right)\right\| < \mu^{-1} < \left\|Df^k\left(v^u\right)\right\|.$$

Under the coordinates given in (1), it can be shown [8] that all partially hyperbolic automorphisms on $\mathcal{H}$ can be represented as

$$f:\ \mathcal{H} \longrightarrow \mathcal{H}, \qquad (x,y,z) \longmapsto \left(A(x,y), \det(A)\cdot z + \psi_{p,q}(x,y)\right).$$

Here $A \in GL(2,\mathbb{Z})$ is a hyperbolic matrix, and

$$\psi_{p,q}(x,y) = \frac{1}{2}acx^2 + \frac{1}{2}bdy^2 + bcxy + \left(\frac{ac}{2}+p\right)x + \left(\frac{bd}{2}+q\right)y,$$

for some $p, q \in \mathbb{Z}$. From this formula, we can see that all partially hyperbolic automorphisms preserve the $\mathbb{S}^1$-fiber structure of $\mathcal{H}$.

More precisely, every $\mathbb{S}^1$-fiber of $\mathcal{H}$ is tangent to the central bundle E^c of f, and the restriction of f on each $\mathbb{S}^1$-fiber is an isometry. Moreover, the sum $E^s \oplus E^u$ of the stable and unstable bundles is a contact plane field on $\mathcal{H}$ which is transverse to each $\mathbb{S}^1$-fiber. See Fig. 1. So our automorphism f is a contactomorphism.

Recently, partially hyperbolic diffeomorphisms on $\mathcal{H}$ are extensively studied, and some beautiful results are achieved. F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures proved that every C^2 volume preserving a partially hyperbolic diffeomorphism on $\mathcal{H}$ is ergodic [7]. Therefore, each partially hyperbolic automorphism on $\mathcal{H}$ is automatically stably ergodic with respect to the Lebesgue volume. It is the topology of $\mathcal{H}$ that guarantees such a persistent mixing property for these partially hyperbolic automorphisms.

After that, A. Hammerlindl and R. Potrie [4,5] showed that each partially hyperbolic diffeomorphism on $\mathcal{H}$ is leaf conjugate to a partially hyperbolic automorphism, i.e., it admits an $\mathbb{S}^1$-fiber structure of $\mathcal{H}$ as its central foliation. Modulo this central foliation, it induces a topological Anosov homeomorphism on a topological torus.

2. Main results

Our original motivation in this work is the famous open problem: *whether the time-1 maps of geodesic flows on surfaces with constant negative curvature are robustly transitive*, see [2,9] for more backgrounds and recent progresses. Note that our partially hyperbolic automorphisms on $\mathcal{H}$ can be regarded as a simplified model for the time-1 maps of geodesic flows on surfaces with constant negative curvature.

Indeed, our partially hyperbolic automorphisms and the time-1 maps of geodesic flows share some common properties: they are both partially hyperbolic; they are both isometries when restricted to the central foliation; and the invariant bundles $E^s \oplus E^u$ for both of them are contact plane fields.

In this note, we show that partially hyperbolic automorphisms on $\mathcal{H}$ are not robustly transitive. This solves *Problem 49* in [6]. More precisely, we show the following:

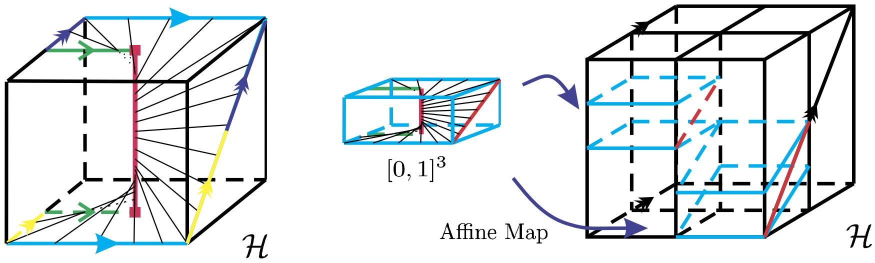

Fig. 2. (Color online.) Left: a Birkhoff section with single boundary fiber. Right: a Birkhoff section with multiple boundary fibers.

Theorem 2.1. *For any partially hyperbolic automorphism $f : \mathcal{H} \longrightarrow \mathcal{H}$, there exists a sequence of C^∞-diffeomorphisms $\{f_n\}$ converging to f in the C^1-topology, such that each f_n is structurally stable, and the chain recurrent set of f_n consists of one attractor and one repeller.*

Remark.

- It is worth to point out that all perturbations f_n are along the $\mathbb{S}^1$-fiber direction, and that f_n keep the same center, center-stable, and center-unstable foliations as f.
- For partially hyperbolic automorphisms on $\mathcal{H}_k$ for $k \geq 2$, the same result holds.

Recall that for a single system, ergodicity implies transitivity. However, it is not known whether stable ergodicity implies robust transitivity. In this note we give the first example of dynamical systems that are stably ergodic, but not robustly transitive. Indeed, while the automorphism f is well known to be stably ergodic, the perturbations f_n of f are not transitive, hence f is not robustly transitive.

Recall that the minimality of stable or unstable foliation of a partially hyperbolic diffeomorphism implies transitivity. This motivates *Problem 50* in [6], which asks *whether minimality of stable and unstable foliations, and accessibility implies robust transitivity for partially hyperbolic diffeomorphisms*. Since our stable and unstable foliations of f are both minimal, and since f is stably accessible [5], this gives a negative answer to *Problem 50*.

Idea of the proof. In the Abelian case of the nilmanifold, an Anosov diffeomorphism times identity on $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$ can be easily perturbed to obtain an attractor and a repeller, since the invariant bundle $E^s \oplus E^u$ is an integrable plane field. Any two integral tori of $E^s \oplus E^u$ could be the attractor and repeller for the new diffeomorphism.

In our case, the topology of $\mathcal{H}$ forbids the existence of any closed surface transverse to the $\mathbb{S}^1$-fiber of $\mathcal{H}$. We need to introduce the Birkhoff section (see [3]) associated with the flow generated by the Reeb vector field of the contact structure $E^s \oplus E^u$, whose flow line is the $\mathbb{S}^1$-fiber structure of $\mathcal{H}$.

Definition 2.2. A Birkhoff section $\Sigma \subset \mathcal{H}$ is an imbedded surface in $\mathcal{H}$ such that the interior of Σ is transverse to the $\mathbb{S}^1$ fibers of $\mathcal{H}$, while the boundary of Σ consists of finitely many $\mathbb{S}^1$ fibers of $\mathcal{H}$ (see Fig. 2).

The central DA-construction [2] allows us to deform the Birkhoff sections into our attractors and repellers. But we need our Birkhoff sections to be invariant. A crucial step of our construction is the following theorem:

Theorem 2.3. *For any partially hyperbolic automorphism $f : \mathcal{H} \longrightarrow \mathcal{H}$, there exists a sequence of Birkhoff sections $\{\Sigma_n\}_{n\geq 1}$ such that $f(\Sigma_n)$ is fiber isotopic to Σ_n. Moreover, far from the boundary fibers, the tangent plane fields of Σ_n converge to the contact plane field $E^s \oplus E^u$ of f as n tends to infinity.*

Remark. In this theorem, fiber isotopy means $f(\Sigma_n)$ could be deformed into Σ_n along the $\mathbb{S}^1$-fiber of $\mathcal{H}$. We call such Birkhoff sections *invariant*. The convergence to the contact plane field means that, for any $\epsilon > 0$, the ratio between the area of the set of points, at which the angle between $T\Sigma_n$ and $E^s \oplus E^u$ is smaller than ϵ, and the area of Σ_n, tends to 1 as n tends to infinity.

Our strategy for proving Theorem 2.1 is to use two parallel invariant Birkhoff sections to approximate the invariant contact structure of f. These two Birkhoff sections will be the candidates for the attractor and the repeller for the new diffeomorphism. Then we apply the central DA-construction [2] to separate these two sections when we are close to their boundary fibers, and to get the attractor and the repeller. □

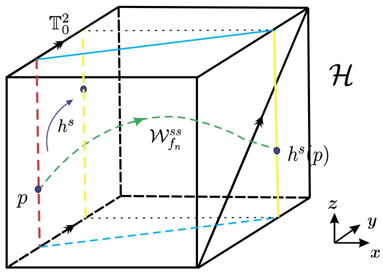

Fig. 3. (Color online.) Holonomy map of stable foliation.

We point out that, for the open problem of time-1 maps of geodesic flows on surfaces with constant negative curvature, if one can find a sequence of Birkhoff sections that approximate their invariant contact structures, then the same techniques can be employed to break the transitivity of the time-1 maps of the geodesic flows.

Complete proofs of the theorems in this section can be found in [8].

3. Applications

An interesting corollary of our theorem is about the holonomy maps of the stable and unstable foliations of the diffeomorphisms $\{f_n\}$.

A homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ is called a *quasiperiodically forced circle homeomorphism* if

$$h(\theta, t) = \big(\theta + \omega, h_\theta(t)\big),$$

where ω is irrational, and the fiber maps h_θ are orientation-preserving circle homeomorphisms. In case h is homotopic to the identity, some examples are constructed in [1], which are transitive but non-minimal quasiperiodically forced circle homeomorphisms. Note that the fiber homeomorphisms h_θ in these examples could be $C^{1+\alpha}$.

The holonomy map of stable and unstable foliations of f_n gives us the following corollary:

Corollary 3.1. *For any $1 \le r < \infty$, there exists a quasiperiodically forced circle homeomorphism*

$$h^r : \mathbb{T}^2 \longrightarrow \mathbb{T}^2, \qquad (\theta, t) \longmapsto \big(\theta + \omega_r, h^r_\theta(t)\big),$$

which is homotopic to Dehn twist, such that h^r is transitive but non-minimal, and each fiber circle homeomorphism h^r_θ is a C^r-diffeomorphism.

Proof. Let us consider the torus $\mathbb{T}^2_0 = \{(0, y, z) : y, z \in [0, 1]\} \subset \mathcal{H}$ by using the coordinates introduced in (1). Then one sees that the stable foliation of f_n is transverse to the torus $\mathbb{T}^2_0$. This induces a holonomy map, say h^s, on $\mathbb{T}^2_0$. Moreover, the $\mathbb{S}^1$-fiber in $\mathbb{T}^2_0$ is the intersection of $\mathbb{T}^2_0$ with the center-stable foliation of f_n. This implies that $h^s : \mathbb{T}^2_0 \to \mathbb{T}^2_0$ is a quasiperiodically forced circle homeomorphism. From the topology of $\mathcal{H}$, h^s is homotopic to a Dehn twist (see Fig. 3).

In [5], it is proved that all the quasiperiodically forced circle homeomorphisms that are homotopic to Dehn's twist are transitive. So does h^s. On the other hand, notice that f_n has a hyperbolic repeller, which must be saturated by the stable leaves of f_n. So the repeller intersects $\mathbb{T}^2_0$ with a proper minimal set. This shows the non-minimality of h^s.

The fiber maps of h^s are the holonomy maps of the strong stable foliation restricted to some center stable manifolds. Since f_n converges to f in the C^1-topology, the norm of the central derivative $\|Df_n|_{E^c}\|$ will converge to 1. Then we apply Theorem 3.2 of [6] to obtain that, for any $1 \le r < \infty$, there exists some n, such that the holonomy map of the strong stable foliation of f_n, when restricted to each center stable manifold, is C^r.

The same analysis works for the unstable foliation and its holonomy map. This finishes the proof of the corollary. □

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