C. R. Acad. Sci. Paris, Ser. I 352 (2014) 719–723

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Algebraic geometry

Normalized non-Archimedean links and surface singularities

Entrelacs non archimédiens normalisés et singularités des surfaces

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ARTICLE INFO

Article history:
Received 24 April 2014
Accepted after revision 29 June 2014
Available online 25 July 2014

Presented by the Editorial Board

ABSTRACT

We define a non-Archimedean analytic version of the link of a singularity, and we use it to study surfaces over an algebraically closed field. This yields a characterization of log essential valuations.

RÉSUMÉ

On définit un analogue en géométrie analytique non archimédienne de l'entrelac d'une singularité, et on l'utilise pour étudier les surfaces sur un corps algébriquement clos. Cela donne une caractérisation des valuations log-essentielles.

Version française abrégée

La théorie de Berkovich donne une structure géométrique à l'espace des valuations qui apparaissent en géométrie birationnelle. Si X est une variété sur un corps non archimédien k, son *analytifié* X^{an} est un espace topologique muni d'une G-topologie et d'un G-faisceau local, le faisceau des *fonctions analytiques*, dont les points peuvent être vus comme des valeurs absolues sur les corps résiduels des points de X.

Lorsque la valeur absolue de k est triviale et Z est une sous-variété de X, Thuillier considère en [10] le sous-espace $\mathrm{L}(X,Z)$ de X^{an} dont les points sont les valeurs absolues sur les corps résiduels de $X \setminus Z$ qui ont un centre en Z. On étudie le quotient de $\mathrm{L}(X,Z)$ par l'action de $\mathbb{R}_{>0}$, qui correspond à une mise à l'échelle des valeurs absolues. On appelle ce quotient, muni aussi de la G-topologie quotient et du poussé en avant par la projection du faisceau des fonctions analytiques, l'*entrelac non archimédien normalisé* de Z dans X, et on le dénote $\mathrm{NL}(X,Z)$. Lorsque $X = \mathbb{A}^2_{\mathbb{C}}$ et $Z = \{0\}$, cette construction généralise l'arbre valuatif de [6], en le munissant d'un faisceau de fonctions analytiques. L'espace $\mathrm{NL}(X,Z)$ est un G-espace localement annelé en k-algèbres. C'est un invariant birationnel de la paire (X,Z) et, si Z contient le lieu singulier de X, alors son type d'homotopie reflète les singularités de X. Sa propriété fondamentale est la suivante.

Théorème 0.1. *En tant que G-espace localement annelé en k-algèbres,* $\mathrm{NL}(X,Z)$ *est localement isomorphe à un espace analytique défini sur* $k((t))$.

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[1] Supported by the Fund for Scientific Research – Flanders (G.0415.10).

http://dx.doi.org/10.1016/j.crma.2014.06.010

La valeur absolue sur $k((t))$ étant une valeur absolue t-adique, ce résultat nous permet d'étudier $\mathrm{NL}(X, Z)$ avec toutes les techniques de la géométrie analytique non archimédienne, y compris celles qui sont seulement valables lorsque la valeur absolue du corps de base n'est pas triviale.

De manière plus générale, on peut définir une catégorie des *espaces normalisés* et caractériser ceux-ci en termes de modèles formels (pour un énoncé précis, voir 2.3).

Ces outils peuvent être appliqués à l'étude des résolutions des surfaces sur un corps algébriquement clos k. Si X est une surface sur k et Z est une sous-variété de X contenant son lieu singulier, le théorème 0.1 implique que $\mathrm{NL}(X, Z)$ est couvert par des $k((t))$-courbes analytiques, et on peut donc se servir des techniques de [5] sur les réductions semi-stables de celles-ci.

À toute modification (Y, D) de la paire (X, Z) dans la catégorie des espaces algébriques telle que D soit un diviseur de Cartier sur Y on associe l'ensemble fini et non vide $\mathrm{Div}(Y)$ de $\mathrm{NL}(X, Z)$ défini comme l'ensemble des $\mathbb{R}_{>0}$-orbites des valuations divisorielles associées aux composantes de D. On dénote $\mathcal{S}$ l'ensemble $\mathrm{Div}(\widetilde{X_Z})$, où $\widetilde{X_Z}$ est l'éclaté normalisé de X en Z. Le résultat suivant stipule que l'on peut construire des modifications avec diviseur exceptionnel donné.

Théorème 0.2. *La correspondance* $(Y, D) \mapsto \mathrm{Div}(Y)$ *est une bijection entre l'ensemble des classes d'isomorphisme des modifications* (Y, D) *de* (X, Z) *dans la catégorie des espaces algébriques avec* Y *normale et* D *diviseur de Cartier sur* Y *et l'ensemble des sous-ensembles finis de valuations divisorielles sur* X *centrées en* Z *contenant* $\mathcal{S}$.

De plus, une modification (Y', D') *en domine une deuxième* (Y, D) *(i.e. il y a un morphisme* $Y' \longrightarrow Y$ *qui commute avec les morphismes vers* X*) si et seulement si* $\mathrm{Div}(Y) \subset \mathrm{Div}(Y')$.

En utilisant une version pour les entrelacs normalisés de [1, 4.3.1] (voir 3.2), on obtient le résultat suivant.

Théorème 0.3. *Soit* (Y, D) *une modification de* (X, Z) *avec* Y *normale et* D *diviseur de Cartier dans* Y*. Alors* Y *est un* k*-schéma lisse et* D *a croisement normaux si et seulement si les composantes connexes de* $\mathrm{NL}(X, Z) \setminus \mathrm{Div}(Y)$ *sont des* $k((t))$*-disques ouverts et un nombre fini de* $k((t))$*-couronnes ouvertes de module* 1.

On remarquera que ceci a du sens, puisque l'on démontre que, pour un sous-espace de $\mathrm{NL}(X, Z)$, la propriété d'être isomorphe à un disque ou à une couronne de module 1, qui sont des sous-espaces de $\mathbb{A}^{1,\mathrm{an}}_{k((t))}$ particulièrement simples, ne dépend pas du choix d'une structure de $k((t))$-espace analytique.

En découpant et recollant des disques et des couronnes, on déduit une caractérisation des *valuations log-essentielles*, i.e. les valuations divisorielles associées à la résolution minimale (Y, D) de (X, Z) telle que D soit un diviseur à croisements normaux de Y:

Théorème 0.4. *Une valuation divisorielle de* X *centrée en* Z *est log-essentielle si et seulement si elle appartient à* $\mathcal{S}$ *ou n'a pas de voisinage dans* $\mathrm{NL}(X, Z) \setminus \mathcal{S}$ *qui est isomorphe à un disque ou à une couronne de module* 1 *et dont le bord topologique est un ensemble de valuations divisorielles.*

1. Introduction

The importance of valuations in the study of resolutions of singularities was emphasized already in the work of Zariski and Abhyankar. Berkovich theory puts a suitable geometric structure on the space of valuations that appear in birational geometry.

If X is a variety defined over a non-Archimedean field, i.e. a field complete with respect to a (possibly trivial) ultrametric absolute value, its Berkovich space X^{an} is a locally ringed space whose points can be seen as real semivaluations, or equivalently as absolute values on residue fields of points of X.

When working over a trivially valued field, Berkovich's theory is a natural framework to study problems in singularity theory. In this context, Thuillier [10] obtained the following result (generalizing a theorem by Stepanov): if X is a variety over a perfect field k, then the homotopy type of the dual graph of the exceptional divisor of a resolution of X does not depend on the choice of the resolution. To prove this, he associates with a subvariety Z of a k-variety X a subspace $\mathrm{L}(X, Z)$ of X^{an}, invariant under modifications of the pair (X, Z), which has the desired homotopy type.

We define a normalized version $\mathrm{NL}(X, Z)$ of $\mathrm{L}(X, Z)$, by taking the quotient of the latter by the action of $\mathbb{R}_{>0}$ corresponding to rescaling semivaluations. If Z is the singular locus of X, $\mathrm{NL}(X, Z)$ can be thought of as a non-Archimedean model of the link of the singularities of X. It is a locally ringed space in k-algebras, endowed with an additional Grothendieck topology, and it can be seen as a generalization of Favre and Jonsson's valuative tree of [6], an object that found many applications in complex dynamics and commutative algebra.

The crucial property of $\mathrm{NL}(X, Z)$ is the following: although $\mathrm{NL}(X, Z)$ is not an analytic space, as a locally ringed space in k-algebras it is locally isomorphic to an analytic space over the field $k((t))$, with a t-adic absolute value. These analytic structures are not canonical.

If X is a surface, this permits to reduce the study of some birational properties of X to the study of smooth analytic curves over the (non-trivially valued) field $k((t))$. We explain how these techniques can give information about the resolutions of singularities of X, and we deduce a characterization of the log essential valuations of X, i.e. the valuations whose center on every log resolution of X is a divisor.

More details and complete proofs will appear in a forthcoming paper of the author.

2. Normalized links

Let k be a non-Archimedean field and let X be a k-variety. The *Berkovich analytification* of X is

$$X^{\mathrm{an}} = \left\{x = (\xi_x, |\ |_x) \text{ s.t. } \xi_x \in X, |\ |_x \text{ absolute value on } \kappa(\xi_x) \text{ extending the absolute value of } k\right\},$$

with the weakest topology such that the map $\rho : X^{\mathrm{an}} \longrightarrow X$ sending a point x to ξ_x is continuous, and for each open U of X and each element f of $\mathcal{O}_X(U)$ the induced map $\rho^{-1}(U) \longrightarrow \mathbb{R}$ sending x to $|f(x)| := |f|_x$ is continuous. Moreover, X^{an} is equipped with a finer G-topology, which in particular keeps track of a family of distinguished compact subspaces, called *affinoid domains*, and the space X^{an} is endowed with a local G-sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ for this G-topology, the *sheaf of analytic functions*. See [1] and [2] for more details about Berkovich spaces.

We assume from now on that the absolute value on k is trivial; for example k might be the field of complex numbers, once endowed with the trivial absolute value. A point x of X^{an} corresponds to a morphism $\mathrm{Spec}(\mathcal{H}(x)) \longrightarrow X$, where $\mathcal{H}(x)$ is the completion of $\kappa(\xi_x)$ with respect to the absolute value $|\ |_x$. If this morphism extends to a morphism from the spectrum of the valuation ring $\mathcal{H}(x)^\circ$ of $\mathcal{H}(x)$ into X, we say that x *has a center* on X, and we define its *center*, denoted by $\mathrm{sp}_X(x)$, as the image of the closed point of $\mathrm{Spec}(\mathcal{H}(x))^\circ$ in X. We obtain a map

$$\mathrm{sp}_X : X^{\beth} := \left\{x \in X^{\mathrm{an}} \;\middle|\; x \text{ has center on } X\right\} \longrightarrow X$$

which is surjective and anticontinuous (i.e. the inverse image of an open subset is closed). Intuitively, if Z is a closed subvariety of X, the open $\mathrm{sp}_X^{-1}(Z)$ of X^{an} can be thought of as an *(infinitesimal) tubular neighborhood* of $Z^{\beth}$ in $X^{\beth}$. Therefore, the subspace $\mathrm{L}(X, Z) := \mathrm{sp}_X^{-1}(Z) \setminus Z^{\beth}$ of X^{an} can be thought of as a *non-Archimedean link* of Z in X.

For example, the underlying topological space of $\mathrm{L}(\mathbb{A}^1_k, 0)$ is homeomorphic to $]0, 1[$: its points are of the form $(\eta, |\ |_\varepsilon)$, where $|\ |_\varepsilon$ is the T-adic absolute value on the residue field $\kappa(\eta) \cong k(T)$ of the affine line at its generic point η such that $|T|_\varepsilon = \varepsilon$, for $\varepsilon \in]0, 1[$. Therefore, the Berkovich space $\mathrm{L}(X, Z)$ distinguishes between equivalent but non-equal absolute values. However, for some applications, it is convenient to identify points corresponding to equivalent absolute values, while keeping as much information as possible about the analytic structure. To do so, we consider the action of the group $\mathbb{R}_{>0}$ on $\mathrm{L}(X, Z)$ defined as follows: if $\lambda \in \mathbb{R}_{>0}$ and $(\xi, |\ |) \in \mathrm{L}(X, Z)$, we set $\lambda \cdot (\xi, |\ |) := (\xi, |\ |^\lambda)$. This defines a free action, and we take the corresponding quotient:

$$\pi : \mathrm{L}(X, Z) \longrightarrow \mathrm{L}(X, Z)/\mathbb{R}_{>0}.$$

Definition 2.1. The *normalized non-Archimedean link* $\mathrm{NL}(X, Z)$ of Z in X is the quotient of $\mathrm{L}(X, Z)$ by the $\mathbb{R}_{>0}$-action defined above, endowed with the quotient topology, the quotient G-topology and the local G-sheaf $\pi_* \mathcal{O}_{\mathrm{L}(X,Z)}$.

Remark 1. The underlying topological space of $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0)$ is homeomorphic to the *valuative tree* of [6], while its sheaf of functions contains much additional information. Moreover, if X is irreducible, $\mathrm{NL}(X, Z)$ can be thought of as a compactification of the normalized valuation space considered in [7]. Indeed, the latter is homeomorphic to the subset of $\mathrm{NL}(X, Z)$ consisting of the points x such that ξ_x is the generic point of X.

The analytic space $\mathrm{L}(X, Z)$ was studied in [10]: it is a birational invariant of the pair (X, Z), and its homotopy type reflects the singularities of X.

We can deduce that the same properties hold for $\mathrm{NL}(X, Z)$: if we call *log modification* of (X, Z) a pair (Y, D) such that Y is a normal algebraic space over k and D is a Cartier divisor of Y, with a proper morphism $f : Y \to X$, which is an isomorphism outside of $D = Y \times_X Z$, then by the valuative criterion of properness such an f induces an isomorphism $\mathrm{NL}(Y, D) \cong \mathrm{NL}(X, Z)$. If moreover Y is a smooth k-scheme and D has normal crossings, we call (Y, D) a *log resolution* of (X, Z); it follows then from [10, 4.7] that $\mathrm{NL}(X, Z)$ has the homotopy type of the dual complex of D.

After taking the quotient of the k-analytic space $\mathrm{L}(X, Z)$ by the $\mathbb{R}_{>0}$-action, we are left with an object that looks like a Berkovich space defined over the Laurent series field $k((t))$ and of dimension one less than the dimension of $\mathrm{L}(X, Z)$. This can be made precise as follows. If X is affine and $Z = V(f)$ is a hypersurface in X, then $|f|$ gives a map from $\mathrm{L}(X, Z)$ to $\mathrm{L}(\mathbb{A}^1_k, 0) \cong]0, 1[$, and it can be proven that, for each $r \in]0, 1[$, π induces an isomorphism of locally ringed G-spaces between $|f|^{-1}(r)$, which as discussed in [2, §4.1] is canonically a $k((t))$-analytic space, and $\mathrm{NL}(X, Z)$. For general (X, Z), we deduce the following result by blowing up X in Z:

Theorem 2.2. *As a locally ringed G-topological space in k-algebras, $\mathrm{NL}(X, Z)$ is locally isomorphic to an analytic space over $k((t))$.*

Since $k((t))$ is a non-Archimedean field with respect to a t-adic valuation, this permits to study $\mathrm{NL}(X, Z)$, and thus deduce information about the pair (X, Z), with tools of non-Archimedean analytic geometry, including the ones that work only over non-trivially valued fields. This result also explains to which extent the valuative tree of [6] looks like an analytic curve over $k((t))$.

Moreover, it can be proven that some properties of a $k((t))$-analytic space depend only on the structure of locally ringed G-space in k-algebras. This is the case for affinoid domains: following [8], it can be shown that a $k((t))$-analytic space V is affinoid if and only if it is a Stein space, quasi-compact and its ring of analytic functions bounded by one is a special k-algebra, i.e. a k-algebra of the form $k[X_1, \dots, X_n][[Y_1, \dots, Y_m]]/I$. Furthermore, by standard arguments of deformation theory of affine formal schemes, the same is true for the properties of being a *disc*, i.e. a subspace of $\mathbb{A}^{1,\mathrm{an}}_{k((t))} = \mathrm{Spec}(k((t))[T])^{\mathrm{an}}$ defined by a condition of the form $\{|T| < 1\}$, or an *annulus of modulus n*, i.e. a subspace of $\mathbb{A}^{1,\mathrm{an}}_{k((t))}$ defined by $\{|t^n| < |T| < 1\}$, $n > 0$. Therefore, it makes sense to speak about affinoid domains, discs and annuli inside $\mathrm{NL}(X, Z)$.

More generally, we can define a category of *normalized spaces* over k. In particular, a normalized space is a quasi-compact locally ringed G-topological space in k-algebras that is covered by affinoid domains. We can then define a functor associating a normalized space $\mathrm{NL}(\mathcal{X})$ with any *special formal k-scheme* $\mathcal{X}$, a quasi-compact formal k-scheme that is locally the formal spectrum of a special k-algebra. It is then possible to characterize categorically the locally ringed spaces that can be obtained in this way, showing a "normalized spaces version" of a classic theorem of Raynaud for non-Archimedean analytic spaces [4, 4.1]:

Theorem 2.3. *The functor* NL *induces an equivalence between the category of special formal k-schemes with adic morphisms, localized by the class of blowups centered in the special fiber, and the category of normalized spaces over k.*

3. Applications to the case of surfaces

The tool explained above can be applied to the study of resolutions of singularities of surfaces. We assume that X is a surface over k and that Z is a closed subvariety of X containing its singular locus, and for simplicity we suppose that the trivially valued field k is algebraically closed. Theorem 2.2 tells us that $\mathrm{NL}(X, Z)$ is covered with $k((t))$-analytic curves, and one can therefore study it using results on the structure of those. In particular, we can make use of the techniques applied in [5] to study semistable reduction for curves using Berkovich geometry.

With each log modification (Y, D) of (X, Z), we can associate a finite non-empty subset $\mathrm{Div}(Y)$ of $\mathrm{NL}(X, Z)$, defined as the set of the $\mathbb{R}_{>0}$-orbits of divisorial valuations associated with the components of D. The set $\mathrm{Div}_{X,Z} := \bigcup_{(Y,D)} \mathrm{Div}(Y)$ is then the set of all $\mathbb{R}_{>0}$-orbits of the divisorial valuations on X centered in Z. We denote by $\mathcal{S}$ the set $\mathrm{Div}(\widetilde{X_Z})$, where $\widetilde{X_Z}$ is the normalization of the blowup of X in Z. The following theorem says that we can construct log modifications with a prescribed exceptional divisor D.

Theorem 3.1. *The correspondence $(Y, D) \mapsto \mathrm{Div}(Y)$ is a bijection between the set of isomorphism classes of log modifications of (X, Z) and the set of finite subsets of $\mathrm{Div}_{X,Z}$ containing $\mathcal{S}$. Moreover, a log modification (Y', D') dominates another one (Y, D) (i.e. there is a morphism $Y' \longrightarrow Y$ commuting with the morphisms to X) if and only if $\mathrm{Div}(Y) \subset \mathrm{Div}(Y')$.*

The hard part of the theorem is the surjectivity of the correspondence. This can be proven by constructing, using the Riemann–Roch theorem, a suitable cover of $\mathrm{NL}(X, Z)$ by affinoid domains, yielding a formal atlas that lets us construct a formal model for $\mathrm{NL}(X, Z)$. This formal k-scheme, once algebrized, gives us the modification we want.

If (Y, D) is a log modification of (X, Z), then $\mathrm{Div}(Y)$ coincides with the set of points of $\mathrm{NL}(X, Z)$ whose center is the generic point of a component of D, and the connected components of $\mathrm{NL}(X, Z) \setminus \mathrm{Div}(Y)$ are precisely the inverse images via sp_Y of the closed points of D. Following Bosch–Lütkebohmert [3, 2.2, 2.3] and Berkovich [1, 4.3.1], we can explicitly describe those inverse images in some good cases.

Proposition 3.2. *Let (Y, D) be a log modification of (X, Z) and let x be a closed point of D. Then:*

(i) *x is smooth both in Y and in D if and only if $\mathrm{sp}_Y^{-1}(x)$ is a disc;*
(ii) *x is smooth in Y and an ordinary double point of D if and only if $\mathrm{sp}_Y^{-1}(x)$ is an annulus of modulus one.*

We deduce the following result:

Corollary 3.3. *A log modification (Y, D) of (X, Z) is a log resolution if and only if the connected components of $\mathrm{NL}(X, Z) \setminus \mathrm{Div}(Y)$ are discs and finitely many annuli of modulus one.*

By carefully cutting and patching discs and annuli, we can characterize the finite set of divisorial valuations corresponding to the minimal log resolution of the pair (X, Z) in terms of the local structure of $\mathrm{NL}(X, Z)$. In turn, we can deduce a characterization of the *log essential valuations* of (X, Z), i.e. the valuations whose center on every log resolution of (X, Z) is a divisor, as follows:

Theorem 3.4. *A point of* $\mathrm{Div}_{X,Z}$ *is a log essential valuation if and only if it belongs to* $\mathcal{S}$ *or it has no neighborhood in* $\mathrm{NL}(X, Z) \setminus \mathcal{S}$ *which is isomorphic to a disc or an annulus of modulus one and whose topological boundary is contained in* $\mathrm{Div}_{X,Z}$.

Note that, even when Z is the singular locus of X, the set of log essential valuations can be larger than the set of essential valuations considered by Nash in [9], since we require the exceptional locus of a resolution to be a normal crossing divisor. However, for many classes of singularities (e.g., rational singularities, quotient singularities) these two notions coincide.

Remark 2. We believe that the approach above can also give a new proof of the existence of the resolution of singularities of the pair (X, Z), when X is a surface over an arbitrary field, by showing the existence of a finite subset S of $\mathrm{Div}_{X,Z}$ such that the connected components of $\mathrm{NL}(X, Z) \setminus S$ are discs and finitely many open annuli. Moreover, if this is the case, then this method could give the minimal log resolution of (X, Z) in only one step. This is object of a work in progress by the author.

Acknowledgement

I warmly thank Johannes Nicaise for his guidance and advice.

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