C. R. Acad. Sci. Paris, Ser. I 352 (2014) 673–677

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Group theory/Geometry

The multiplicities of the equivariant index of twisted Dirac operators

Multiplicités de l'indice équivariant de l'opérateur de Dirac twisté

Paul-Émile Paradan, Michèle Vergne

ARTICLE INFO

Article history:
Received 14 April 2014
Accepted 7 May 2014
Available online 14 August 2014

Presented by Michèle Vergne

ABSTRACT

In this note, we give a geometric expression for the multiplicities of the equivariant index of a Dirac operator twisted by a line bundle.

RÉSUMÉ

Le but de cette note est de donner une expression géométrique pour les multiplicités de l'indice équivariant de l'opérateur de Dirac tordu par un fibré en lignes.

1. Introduction

Let M be a compact connected manifold. We assume that M is even dimensional and oriented. We consider a spin^c structure on M, and denote by $\mathcal{S}$ the corresponding irreducible Clifford module. Let K be a compact connected Lie group acting on M, and preserving the spin^c structure. We denote by $D : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-)$ the corresponding twisted Dirac operator. The equivariant index of D, denoted $\mathrm{Q}_K^{\mathrm{spin}}(M)$, belongs to the Grothendieck group of representations of K,

$$\mathrm{Q}_K^{\mathrm{spin}}(M) = \sum_{\pi \in \widehat{K}} \mathrm{m}(\pi)\pi .$$

An important example is when M is a compact complex manifold, K a compact group of holomorphic transformations of M, and $\mathcal{L}$ any holomorphic K-equivariant line bundle on M (not necessarily ample). Then the Dolbeault operator twisted by $\mathcal{L}$ can be realized as a twisted Dirac operator D. In this case $\mathrm{Q}_K^{\mathrm{spin}}(M) = \sum_q (-1)^q H^{0,q}(M, \mathcal{L})$.

The aim of this note is to give a geometric description of the multiplicity $\mathrm{m}(\pi)$ in the spirit of the Guillemin–Sternberg phenomenon $[Q, R] = 0$ [5,7,8,11,9].

Consider the determinant line bundle $\mathbb{L} = \det(\mathcal{S})$ of the spin^c structure. This is a K-equivariant complex line bundle on M. The choice of a K-invariant Hermitian metric and of a K-invariant Hermitian connection ∇ on $\mathbb{L}$ determines an abstract moment map

$$\Phi_\nabla : M \to \mathfrak{k}^*$$

E-mail addresses: Paul-Emile.Paradan@univ-montp2.fr (P.-É. Paradan), Michele.Vergne@math.jussieu.fr (M. Vergne).

http://dx.doi.org/10.1016/j.crma.2014.05.001

by the relation $2i\langle\Phi_\nabla, X\rangle$, for all $X \in \mathfrak{k}$. We compute $\mathrm{m}(\pi)$ in term of the reduced "manifolds" $\Phi_\nabla^{-1}(f)/K_f$. This formula extends the result of [10]. However, in this note, we do not assume any hypothesis on the line bundle $\mathbb{L}$, in particular we do not assume that the curvature of the connection ∇ is a symplectic form. In this pre-symplectic setting, a (partial) answer to this question has been obtained by [6,3,4,1] when K is a torus. Our method is based on localization techniques as in [9,10].

2. Admissible coadjoints orbits

We consider a compact connected Lie group K with Lie algebra $\mathfrak{k}$. Consider an admissible coadjoint orbit $\mathcal{O}$ (as in [2]), oriented by its symplectic structure. Then $\mathcal{O}$ carries a K-equivariant bundle of spinors $\mathcal{S}_\mathcal{O}$, such that the associated moment map is the injection $\mathcal{O}$ in $\mathfrak{k}^*$. We denote by $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$ the corresponding equivariant index.

Let us describe the admissible coadjoint orbits with their spin^c index.

Let T be a Cartan subgroup of K with Lie algebra $\mathfrak{t}$. Let $\Lambda \subset \mathfrak{t}^*$ be the lattice of weights of T (thus $\mathrm{e}^{i\lambda}$ is a character of T). Choose a positive system $\Delta^+ \subset \mathfrak{t}^*$, and let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Let $\mathfrak{t}^*_{\geq 0}$ be the closed Weyl chamber and we denote by $\mathcal{F}$ the set of the relative interiors of the faces of $\mathfrak{t}^*_{\geq 0}$. Thus $\mathfrak{t}^*_{\geq 0} = \coprod_{\sigma\in\mathcal{F}}\sigma$, and we denote $\mathfrak{t}^*_{>0} \in \mathcal{F}$ the interior of $\mathfrak{t}^*_{\geq 0}$.

We index the set $\hat{K}$ of classes of finite dimensional irreducible representations of K by the set $(\Lambda+\rho)\cap\mathfrak{t}^*_{>0}$. The irreducible representation π_λ corresponding to $\lambda \in (\Lambda+\rho)\cap\mathfrak{t}^*_{>0}$ is the irreducible representation with infinitesimal character λ. Its highest weight is $\lambda-\rho$.

Let $\sigma \in \mathcal{F}$. The stabilizer K_ξ of a point $\xi \in \sigma$ depends only of σ. We denote it by K_σ, and by $\mathfrak{k}_\sigma$ its Lie algebra. We choose on $\mathfrak{k}_\sigma$ the system of positive roots contained in Δ^+, and let ρ_σ be the corresponding ρ.

When $\mu \in \sigma$, the coadjoint orbit $K\cdot\mu$ is admissible if and only if $\mu-\rho+\rho_\sigma \in \Lambda$. The spin^c equivariant index of the admissible orbits is described in the following lemma.

Lemma 2.1. *Let $K\cdot\mu$ be an admissible orbit: $\mu\in\sigma$ and $\mu-\rho+\rho_\sigma\in\Lambda$. If $\mu+\rho_\sigma$ is regular, then $\mu+\rho_\sigma \in \rho+\overline{\sigma}$. Thus we have:*

$$\mathrm{Q}_K^{\mathrm{spin}}(K\cdot\mu) = \begin{cases} 0 & \text{if } \mu+\rho_\sigma \text{ is singular},\\ \pi_{\mu+\rho_\sigma} & \text{if } \mu+\rho_\sigma \text{ is regular}. \end{cases}$$

In particular, if $\lambda \in (\Lambda+\rho)\cap\mathfrak{t}^*_{>0}$, then $K\cdot\lambda$ is admissible and $\mathrm{Q}_K^{\mathrm{spin}}(K\cdot\lambda) = \pi_\lambda$.

Let $\mathcal{H}_\mathfrak{k}$ be the set of conjugacy classes of the reductive algebras $\mathfrak{k}_f$, $f\in\mathfrak{k}^*$. We denote by $\mathcal{S}_\mathfrak{k}$ the set of conjugacy classes of the semi-simple parts $[\mathfrak{h},\mathfrak{h}]$ of the elements $(\mathfrak{h})\in\mathcal{H}_\mathfrak{k}$. The map $(\mathfrak{h})\to([\mathfrak{h},\mathfrak{h}])$ induces a bijection between $\mathcal{H}_\mathfrak{k}$ and $\mathcal{S}_\mathfrak{k}$.

The map $\mathcal{F}\longrightarrow\mathcal{H}_\mathfrak{k}$, $\sigma\mapsto(\mathfrak{k}_\sigma)$, is surjective and for $(\mathfrak{h})\in\mathcal{H}_\mathfrak{k}$ we denote by

- $\mathcal{F}(\mathfrak{h})$ the set of $\sigma\in\mathcal{F}$ such that $(\mathfrak{k}_\sigma) = (\mathfrak{h})$,
- $\mathfrak{k}^*_\mathfrak{h} \subset \mathfrak{k}^*$ the set of elements $f\in\mathfrak{k}^*$ with infinitesimal stabilizer $\mathfrak{k}_f$ belonging to the conjugacy class $(\mathfrak{h})$.

We have $\mathfrak{k}^*_\mathfrak{h} = K(\bigcup_{\sigma\in\mathcal{F}(\mathfrak{h})}\sigma)$. In particular, all coadjoint orbits contained in $\mathfrak{k}^*_\mathfrak{h}$ have the same dimension. We say that such a coadjoint orbit is of type $(\mathfrak{h})$. If $(\mathfrak{h}) = (\mathfrak{t})$, then $\mathfrak{k}^*_\mathfrak{h}$ is the open subset of regular elements.

We denote by $A(\mathfrak{h})$ the set of admissible coadjoint orbits of type $(\mathfrak{h})$. This is a discrete subset of orbits in $\mathfrak{k}^*_\mathfrak{h}$.

Example 1. Consider the group $K = SU(3)$ and let $(\mathfrak{h})$ be the conjugacy class such that $\mathfrak{k}^*_\mathfrak{h}$ is equal to the set of sub-regular elements $f\in\mathfrak{k}^*$ (the orbit of f is of dimension $\dim(K/T)-2$). Let ω_1,ω_2 be the two fundamental weights. Let σ_1,σ_2 be the half lines $\mathbb{R}_{>0}\omega_1$, $\mathbb{R}_{>0}\omega_2$. Then $\mathfrak{k}^*_\mathfrak{h}\cap\mathfrak{t}^*_{\geq 0} = \sigma_1\cup\sigma_2$. The set $A(\mathfrak{h})$ is equal to the collection of orbits $K\cdot(\frac{1+2n}{2}\omega_i)$, $n\in\mathbb{Z}_{\geq 0}$, $i=1,2$. The representation $\mathrm{Q}_K^{\mathrm{spin}}(K\cdot(\frac{1+2n}{2}\omega_i))$ is 0 if $n=0$, otherwise it is the irreducible representation $\pi_{\rho+(n-1)\omega_i}$. In particular, both representations associated with the admissible orbits $\frac{3}{2}\omega_1$ and $\frac{3}{2}\omega_2$ are the trivial representation π_ρ.

3. The theorem

Consider the action of K in M. Let $(\mathfrak{k}_M)$ be the conjugacy class of the generic infinitesimal stabilizer. On a K-invariant open and dense subset of M, the conjugacy class of $\mathfrak{k}_m$ is equal to $(\mathfrak{k}_M)$. Consider the (conjugacy class) $([\mathfrak{k}_M,\mathfrak{k}_M])$.

We start by stating two vanishing lemmas.

Lemma 3.1. *If $([\mathfrak{k}_M,\mathfrak{k}_M])$ does not belong to the set $\mathcal{S}_\mathfrak{k}$, then $\mathrm{Q}_K^{\mathrm{spin}}(M) = 0$ for any K-invariant spin^c structure on M.*

If $([\mathfrak{k}_M,\mathfrak{k}_M]) = ([\mathfrak{h},\mathfrak{h}])$ for some $(\mathfrak{h})\in\mathcal{H}_\mathfrak{k}$, any K-invariant map $\Phi: M\to\mathfrak{k}^*$ is such that $\Phi(M)$ is included in the closure of $\mathfrak{k}^*_\mathfrak{h}$.

Lemma 3.2. *Assume that* $([\mathfrak{k}_M,\mathfrak{k}_M]) = ([\mathfrak{h},\mathfrak{h}])$ *with* $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$. *Let us consider a* spin^c *structure on* M *with determinant bundle* $\mathbb{L}$. *If there exists a* K*-invariant Hermitian connection* ∇ *on* $\mathbb{L}$ *such that* $\Phi_\nabla(M) \cap \mathfrak{k}^*_\mathfrak{h} = \emptyset$, *then* $\mathrm{Q}^{\mathrm{spin}}_K(M) = 0$.

Thus from now on, we assume that the action of K on M is such that $([\mathfrak{k}_M,\mathfrak{k}_M]) = ([\mathfrak{h},\mathfrak{h}])$ for some $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$. Let us consider a spin^c structure on M with determinant bundle $\mathbb{L}$ and a K-invariant Hermitian connection with moment map $\Phi_\nabla : M \to \mathfrak{k}^*$.

We extend the definition of the index to disconnected even dimensional oriented manifolds by defining $\mathrm{Q}^{\mathrm{spin}}_K(M)$ to be the sum over the connected components of M. If K is the trivial group, $\mathrm{Q}^{\mathrm{spin}}_K(M) \in \mathbb{Z}$ and is denoted simply by $\mathrm{Q}^{\mathrm{spin}}(M)$.

Consider a coadjoint orbit $\mathcal{O} = K \cdot f$. The reduced space $M_\mathcal{O}$ is defined to be the topological space $\Phi_\nabla^{-1}(\mathcal{O})/K = \Phi_\nabla^{-1}(f)/K_f$. We also denote it by M_f. This space might not be connected.

In the next section, we define a $\mathbb{Z}$-valued function $\mathcal{O} \mapsto \mathrm{Q}^{\mathrm{spin}}(M_\mathcal{O})$ on the set $A(\mathfrak{h})$ of admissible orbits of type $(\mathfrak{h})$. We call it the reduced index:

- if $M_\mathcal{O} = \emptyset$, then $\mathrm{Q}^{\mathrm{spin}}(M_\mathcal{O}) = 0$,
- when $M_\mathcal{O}$ is an orbifold, the reduced index $\mathrm{Q}^{\mathrm{spin}}(M_\mathcal{O})$ is defined as an index of a Dirac operator associated with a natural "reduced" spin^c structure on $M_\mathcal{O}$.

Otherwise, it is defined via a limit procedure. Postponing this definition, we have the following theorem.

Theorem 3.3. *Assume that* $([\mathfrak{k}_M,\mathfrak{k}_M]) = ([\mathfrak{h},\mathfrak{h}])$ *with* $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$. *Then*

$$\mathrm{Q}^{\mathrm{spin}}_K(M) = \sum_{\mathcal{O}\in A(\mathfrak{h})} \mathrm{Q}^{\mathrm{spin}}(M_\mathcal{O})\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{O}).$$

In the expression above, when $(\mathfrak{h})$ is not Abelian, $\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{O})$ can be 0, and several orbits $\mathcal{O} \in A(\mathfrak{h})$ can give the same representation.

Theorem 3.3 is in the spirit of the $[Q,R]=0$ theorem. However, it has some radically new features. First, as Φ_∇ is not the moment map of a Hamiltonian structure, the definition of the reduced space requires more care. For example, the fibers of Φ_∇ might not be connected, and the Kirwan set $\Phi_\nabla(M) \cap \mathfrak{t}^*_{\geq 0}$ is not a convex polytope. Furthermore, this Kirwan set depends on the choice of connection ∇. Second, the map $\mathcal{O} \in A(\mathfrak{h}) \to \mathrm{Q}^{\mathrm{spin}}_K(\mathcal{O})$ is not injective, when $\mathfrak{h}$ is not Abelian. Thus the multiplicities m_λ of the representation π_λ in $\mathrm{Q}^{\mathrm{spin}}_K(M)$ will be eventually obtained as a sum of reduced indices involving several reduced spaces.

We explicit this last point.

Theorem 3.4. *Assume that* $([\mathfrak{k}_M,\mathfrak{k}_M]) = ([\mathfrak{h},\mathfrak{h}])$ *with* $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$. *Let* $\mathrm{m}_\lambda \in \mathbb{Z}$ *be the multiplicity of the representation* π_λ *in* $\mathrm{Q}^{\mathrm{spin}}_K(M)$. *We have:*

$$\mathrm{m}_\lambda = \sum_{\substack{\sigma\in\mathcal{F}(\mathfrak{h})\\ \lambda-\rho_\sigma\in\sigma}} \mathrm{Q}^{\mathrm{spin}}(M_{\lambda-\rho_\sigma}). \tag{1}$$

More explicitly, the sum is taken over the (relative interiors of) faces σ of the Weyl chamber such that:

$$\big([\mathfrak{k}_M,\mathfrak{k}_M]\big) = \big([\mathfrak{k}_\sigma,\mathfrak{k}_\sigma]\big), \qquad \Phi_\nabla(M)\cap\sigma \neq \emptyset, \qquad \lambda \in \{\sigma + \rho_\sigma\}. \tag{2}$$

If $\mathfrak{k}_M$ is Abelian, we have simply $\mathrm{m}_\lambda = \mathrm{Q}^{\mathrm{spin}}(\Phi_\nabla^{-1}(\lambda)/T)$. In particular, if the group K is the circle group, and λ is a regular value of the moment map Φ_∇, this result was obtained in [1].

If $\mathfrak{k}_M$ is not Abelian, and the curvature of the connection ∇ is symplectic, Kirwan's convexity theorem implies that the image $\Phi_\nabla(M) \cap \mathfrak{t}^*_{\geq 0}$ is contained in the closure of one single σ. Thus there is a unique σ satisfying Conditions (2). In this setting, Theorem 3.4 is obtained in [10].

Let us give an example where several σ contribute to the multiplicity of a representation π_λ.

We take the notations of Example 1. We label ω_1,ω_2 so that $\mathfrak{k}_{\omega_1}$ is the group $S(U(2)\times U(1))$ stabilizing the line $\mathbb{C}e_3$ in the fundamental representation of $SU(3)$ in $\mathbb{C}^3 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$.

Let $P = \{0 \subset L_2 \subset L_3 \subset \mathbb{C}^4\}$ be the partial flag manifold with L_2 a subspace of $\mathbb{C}^4$ of dimension 2 and L_3 a subspace of $\mathbb{C}^4$ of dimension 3. Denote by $\mathcal{L}_1, \mathcal{L}_2$ the equivariant line bundles on P with fiber at (L_2, L_3) the one-dimensional spaces $\wedge^2 L_2$ and L_3/L_2, respectively. Let M be the subset of P where L_2 is assumed to be a subspace of $\mathbb{C}^3$. Thus M is fibered over $P_2(\mathbb{C})$ with fiber $P_1(\mathbb{C})$. The group $SU(3)$ acts naturally on M, and the generic stabilizer of the action is $SU(2)$. We denote by $\mathcal{L}_{a,b}$ the line bundle $\mathcal{L}_1^a \otimes \mathcal{L}_2^b$ restricted to M. This line bundle is equipped with a natural holomorphic and Hermitian connection ∇. Consider the spin^c structure with determinant bundle $\mathbb{L} = \mathcal{L}_{2a+1,2b+1}$, where a, b are positive integers. If $a \geq b$, the curvature of the line bundle $\mathbb{L}$ is non-degenerate, and we are in the symplectic case. Let us consider $b > a$. It is easy to see that, in this case, the Kirwan set $\Phi_\nabla(M) \cap \mathfrak{t}^*_{\geq 0}$ is the non-convex set $[0, b-a]\omega_1 \cup [0, a+1]\omega_2$. We compute the character of the representation $\mathrm{Q}^{\mathrm{spin}}_K(M)$ by the Atiyah–Bott fixed point formula, and find:

$$\mathrm{Q}^{\mathrm{spin}}_K(M) = \sum_{j=0}^{b-a-2} \pi_{\rho + j\omega_1} \oplus \sum_{j=0}^{a-1} \pi_{\rho + j\omega_2}.$$

In particular, the multiplicity of π_ρ (the trivial representation) is equal to 2. We use now Theorem 3.3 and the discussion of Example 1, and obtain (reduced multiplicities are equal to 1):

$$\mathrm{Q}^{\mathrm{spin}}_K(M) = \sum_{j=0}^{b-a-1} \mathrm{Q}^{\mathrm{spin}}_K\left(K \cdot \left(\frac{1+2j}{2}\omega_1\right)\right) \oplus \sum_{j=0}^{a} \mathrm{Q}^{\mathrm{spin}}_K\left(K \cdot \left(\frac{1+2j}{2}\omega_2\right)\right).$$

Using the formulae for $\mathrm{Q}^{\mathrm{spin}}_K(K \cdot (\frac{1+2n}{2}\omega_i))$ given in Example 1, these two formulae (fortunately) coincide. Furthermore we see that both faces σ_1, σ_2 give a non-zero contribution to the multiplicity of the trivial representation.

4. Definition of the reduced index

We start by defining the reduced index for the action of an Abelian torus H on a connected manifold Y. Denote by Λ the lattice of weights of H. We do not assume Y compact, but we assume that the set of stabilizers H_m of points in Y is finite. Let $\mathfrak{h}_Y$ be the generic infinitesimal stabilizer of the action H on Y, and H_Y be the connected subgroup of H with Lie algebra $\mathfrak{h}_Y$. Thus H_Y acts trivially on Y. Let us consider a spin^c structure on Y with determinant bundle $\mathbb{L}$, and an H-invariant connection ∇ on $\mathbb{L}$. The image $\Phi_\Delta(Y)$ spans an affine space I_Y parallel to $\mathfrak{h}_Y^\perp$. We assume that the fibers of the map Φ_Δ are compact. We can easily prove that there exists a finite collection of hyperplanes $W^1, \ldots, W^p$ in I_Y such that the group H/H_Y acts locally freely on $\Phi_\Delta^{-1}(f)$, when f is in $\Phi_\nabla(Y)$, but not on any of the hyperplanes W^i.

Proposition 1.

- *When $\mu \in I_Y \cap \Lambda$ is a regular value of $\Phi_\nabla : Y \to I_Y$, the reduced space Y_μ is an oriented orbifold equipped with an induced* spin^c *structure: we denote $\mathrm{Q}^{\mathrm{spin}}(Y_\mu)$ the corresponding* spin^c *index.*
- *With any connected component $\mathcal{C}$ of $I_Y \setminus \bigcup_{k=1}^p W^k$, we can associate a periodic polynomial function $q^{\mathcal{C}} : \Lambda \cap I_Y \to \mathbb{Z}$ such that*

$$q^{\mathcal{C}}(\mu) = \mathrm{Q}^{\mathrm{spin}}(Y_\mu)$$

for any element $\mu \in \Lambda \cap \mathcal{C}$ which is a regular value of $\Phi : Y \to I_Y$.
- *If $\mu \in \Lambda$ belongs to the closure of two connected components $\mathcal{C}_1$ and $\mathcal{C}_2$ of $I_Y \setminus \bigcup_{k=1}^p W^k$, we have:*

$$q^{\mathcal{C}_1}(\mu) = q^{\mathcal{C}_2}(\mu).$$

We can now state the definition of the "reduced" index on Λ:

- $\mathrm{Q}^{\mathrm{spin}}(Y_\mu) = 0$ if $\mu \notin \Lambda \cap I_Y$,
- for any $\mu \in \Lambda \cap I_Y$, we define $\mathrm{Q}^{\mathrm{spin}}(Y_\mu)$ as being equal to $q^{\mathcal{C}}(\mu)$, where $\mathcal{C}$ is any connected component containing μ in its closure. In fact, $\mathrm{Q}^{\mathrm{spin}}(Y_\mu)$ is computed as an index of a particular spin^c structure on the orbifold $\Phi_\nabla^{-1}(\mu + \epsilon)/H$ for any ϵ small and such that $\mu + \epsilon$ is a regular value of Φ_∇.

If Y is not connected, we define the reduced index at a point $\mu \in \Lambda$ as the sum of reduced indices over all connected components of Y.

More generally, let H be a compact connected group acting on Y and such that $[H, H]$ acts trivially on Y. Let $\mathcal{S}_Y$ be an equivariant spin^c structure on Y with determinant bundle $\mathbb{L}$. For any $\mu \in \mathfrak{h}^*$ such that $\mu([\mathfrak{h}, \mathfrak{h}]) = 0$, and admissible for H, it is then possible to define $\mathrm{Q}^{\mathrm{spin}}(Y_\mu)$. Indeed eventually passing to a double cover of the torus $H/[H, H]$, and translating by the square root of the action of $H/[H, H]$ on the fiber of $\mathbb{L}$, we are reduced to the preceding case of the action of the torus $H/[H, H]$, and an $H/[H, H]$-equivariant spin^c structure on Y.

Consider now the action of a connected compact group K on M. Let σ be a (relative interior) of a face of $\mathfrak{t}^*_{\geq 0}$ that satisfies the following conditions:

$$([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]), \qquad \Phi_\nabla^{-1}(\sigma) \neq \emptyset. \tag{3}$$

Let us explain how to compute the "reduced" index map $\mu \to \mathrm{Q}^{\mathrm{spin}}(M_\mu)$ on the set $\sigma \cap \{\Lambda + \rho - \rho_\sigma\}$ that parameterizes the admissible orbits intersecting σ.

We work with the "slice" Y defined by σ. The set $U_\sigma := K_\sigma(\bigcup_{\sigma\subset\bar{\tau}}\tau)$ is an open neighborhood of σ in $\mathfrak{k}^*_\sigma$ such that the open subset $KU_\sigma \subset \mathfrak{k}^*$ is isomorphic to $K \times_{K_\sigma} U_\sigma$. We consider the K_σ-invariant subset $Y = \Phi_\nabla^{-1}(U_\sigma)$. The following lemma allows us to reduce the problem to the Abelian case.

Lemma 4.1.

- *Y is a non-empty submanifold of M such that KY is an open subset of M isomorphic to $K \times_{K_\sigma} Y$.*
- *The Clifford module $\mathcal{S}_M$ on M determines a Clifford module $\mathcal{S}_Y$ on Y with determinant line bundle $\mathbb{L}_Y = \mathbb{L}_M|_Y \otimes \mathbb{C}_{-2(\rho-\rho_\sigma)}$. The corresponding moment map is $\Phi_\nabla|_Y - \rho + \rho_\sigma$.*
- *The group $[K_\sigma, K_\sigma]$ acts trivially on Y and on the bundle of spinors $\mathcal{S}_Y$.*

We thus consider Y with action of K_σ, and Clifford bundle $\mathcal{S}_Y$. If $\mu \in \sigma$ is admissible for K, then $\mu - \rho + \rho_\sigma \in \Lambda$ is admissible for K_σ. The reduced space $M_\mu = \Phi_\nabla^{-1}(\mu)/K_\sigma$ is equal to the reduced space $Y_{\mu-\rho+\rho_\sigma}$. As $[K_\sigma, K_\sigma]$ acts trivially on $(Y, \mathcal{S}_Y)$, we are in the Abelian case, and we define $\mathrm{Q}^{\mathrm{spin}}(M_\mu) := \mathrm{Q}^{\mathrm{spin}}(Y_{\mu-\rho+\rho_\sigma})$.

Acknowledgements

We wish to thank the Research in Pairs program at Mathematisches Forschungsinstitut Oberwolfach (January 2014), which gave us the opportunity to work on these questions.

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