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Number theory

On the cohomology of semi-stable p -adic Galois representations



Sur la cohomologie des représentations galoisiennes p -adiques semi-stables

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ABSTRACT

Let K be a field of characteristic 0 complete with respect to a non-trivial discrete valuation with perfect residue field k of characteristic $p > 0$. Let V be a p -adic representation of the absolute Galois group of K . We compute explicitly Kato's filtration on the continuous cohomology group $H^1(K, V)$. When k is finite, we give a simple proof of Hyodo's celebrated result $H_g^1(K, V) = H_{st}^1(K, V)$ when V is a potentially semi-stable Galois representation.

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R É S U M É

Soit K un corps de caractéristique 0 complet pour une valuation discrète non triviale à corps résiduel parfait k de caractéristique $p > 0$. Soit V une représentation p -adique du groupe de Galois absolu de K . On calcule explicitement la filtration de Kato sur le groupe de cohomologie continue $H^1(K, V)$. Lorsque k est fini, on en déduit une preuve simple du résultat bien connu de Hyodo qui dit que, si V est potentiellement semi-stable, alors $H_g^1(K, V) = H_{st}^1(K, V)$.

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1. Explicit computation of Kato's filtration on the Galois cohomology

We fix a prime number p and a perfect field k of characteristic $p > 0$. We denote K_0 the fraction field of Witt vectors with coefficients in k and we fix a finite totally ramified extension K of K_0 . We choose an algebraic closure \bar{K} of K and set $G_K = \text{Gal}(\bar{K}/K)$.

The topological \mathbb{Q}_p -vector spaces V equipped with a linear and continuous action of G_K form, in an obvious way, a \mathbb{Q}_p -linear additive exact category $C_{\mathbb{Q}_p}(G_K)$. For any object V of this category and $i \in \mathbb{N}$, we denote $H^i(K, V) = H_{\text{cont}}^i(G_K, V)$ the i -th group of continuous cohomology (see Tate [7, §2]). Given a short exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

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of $C_{\mathbb{Q}_p}(G_K)$, we have an obvious exact sequence¹

$$0 \rightarrow H^0(K, V') \rightarrow H^0(K, V) \rightarrow H^0(K, V'') \rightarrow H^1(K, V') \rightarrow H^1(K, V) \rightarrow H^1(K, V'')$$

With the extension \bar{K}/K are associated the p -adic completion C of \bar{K} and the usual rings of p -adic periods B_{dR} , B_{cris} and B_{st} which are topological rings equipped with a \mathbb{Q}_p -linear and continuous action of $G_K = \text{Gal}(\bar{K}/K)$ (cf. [3] or [4]).

Let's choose a non-zero topologically nilpotent element π of K and a sequence $\varpi = (\varpi^{(n)})_{n \in \mathbb{N}}$ of elements of \bar{K} such that $\varpi^{(0)} = \pi$ and $(\varpi^{(n+1)})^p = \varpi^{(n)}$ for all $n \in \mathbb{N}$. Recall that this choice defines an element $u = \log[\varpi]$ of B_{st} and that we can view also u as an element of B_{dR} by deciding that $\log(\pi) = 0$ (then we identify u to $\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{(u-1)^n}{n\pi^n}$). With these choices,

$$B_{st} = B_{cris}[u]$$

is a polynomial algebra in u with coefficients in B_{cris} and is a G_K -stable subring of B_{dR} . Moreover B_{dR} is a field containing K and, if we denote K_0 the fraction field of the ring $W(k)$ of Witt vectors with coefficients in k , we have:

$$H^0(K, B_{dR}^+) = H^0(K, B_{dR}) = K \quad \text{and} \quad H^0(K, B_{cris}) = H^0(K, B_{st}) = K_0.$$

The ring B_{st} is equipped with an endomorphism φ semi-linear with respect to the absolute Frobenius on K_0 and the B_{cris} -derivation $N = -d/du$. The operators φ and N commute with G_K and satisfy $N\varphi = p\varphi N$. Therefore B_{cris} is the subring of B_{st} kernel of N and we define the ring B_e as the subring of B_{cris} , which is fixed by $\varphi - 1$. We have short exact sequences:

$$\begin{aligned} 0 &\longrightarrow B_{cris} \longrightarrow B_{st} \xrightarrow{N} B_{st} \longrightarrow 0, \\ 0 &\longrightarrow B_e \longrightarrow B_{cris} \xrightarrow{\varphi-1} B_{cris} \longrightarrow 0. \end{aligned} \tag{1}$$

We set $\tilde{B}_{dR} = B_{dR}/B_{dR}^+$ and, for all $b \in B_{dR}$, we denote \tilde{b} its image in \tilde{B}_{dR} . The *fundamental exact sequence of p -adic Hodge theory* is the exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow \tilde{B}_{dR} \longrightarrow 0 \tag{3}$$

where $B_e \mapsto \tilde{B}_{dR}$ is the compositum of the inclusion $B_e \subset B_{dR}$ with the projection $B_{dR} \rightarrow \tilde{B}_{dR}$.

We now consider a p -adic Galois representation, i.e. a finite-dimensional \mathbb{Q}_p -vector space V equipped with a continuous linear action of G_K . Recall that we have a natural filtration by sub- \mathbb{Q}_p -vector spaces on $H^1(K, V)$, the *Kato's filtration*:

$$0 \subset H_e^1(K, V) \subset H_f^1(K, V) \subset H_{st}^1(K, V) \subset H_g^1(K, V) \subset H^1(K, V)$$

where

$$\begin{aligned} H_e^1(K, V) &= \ker(H^1(K, V) \longrightarrow H^1(K, B_e \otimes_{\mathbb{Q}_p} V)), \\ H_f^1(K, V) &= \ker(H^1(K, V) \longrightarrow H^1(K, B_{cris} \otimes_{\mathbb{Q}_p} V)), \\ H_{st}^1(K, V) &= \ker(H^1(K, V) \longrightarrow H^1(K, B_{st} \otimes_{\mathbb{Q}_p} V)), \\ H_g^1(K, V) &= \ker(H^1(K, V) \longrightarrow H^1(K, B_{dR} \otimes_{\mathbb{Q}_p} V)). \end{aligned}$$

We want to compute these cohomology groups. Recall that [5, Chap. I, §2.2.1] the *tangent space* of V is the K -vector space:

$$t_V = H^0(K, \tilde{B}_{dR} \otimes V).$$

We let N and φ act on $B_{st} \otimes_{\mathbb{Q}_p} V$ via $N(b \otimes v) = Nb \otimes v$ and $\varphi(b \otimes v) = \varphi b \otimes v$. These actions commute with the action of G_K , hence N and φ act also on

$$D = D_{st}(V) = H^0(K, B_{st} \otimes_{\mathbb{Q}_p} V)$$

which is a finite-dimensional K_0 -vector space.

¹ If there is a (set-theoretic) continuous splitting of the projection $V \rightarrow V''$, we even get the usual long exact sequence (loc. cit.), but we will not use this fact.

1.1. $H_e^1(K, V)$

Tensoring with V , we get from (3) a short exact sequence

$$0 \longrightarrow V \longrightarrow B_e \otimes V \longrightarrow \tilde{B}_{dR} \otimes V \longrightarrow 0$$

inducing a long exact sequence

$$0 \longrightarrow H^0(K, V) \longrightarrow D_{N=0, \varphi=1} \longrightarrow t_V \longrightarrow H_e^1(K, V) \longrightarrow 0 \tag{S_e}$$

where

$$D_{N=0, \varphi=1} = H^0(K, B_e \otimes V) = \{x \in D \mid Nx = 0, \varphi(x) = x\}.$$

1.2. $H_f^1(K, V)$

Consider the map $B_{cris} \rightarrow B_{cris} \oplus \tilde{B}_{dR}$ sending b to $(\varphi b - b, \tilde{b})$. From the exactness of (2) and (3), we get the exactness of

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{cris} \longrightarrow B_{cris} \oplus \tilde{B}_{dR} \longrightarrow 0. \tag{4}$$

Tensoring with V , we get a short exact sequence

$$0 \longrightarrow V \longrightarrow B_{cris} \otimes V \longrightarrow (B_{cris} \otimes V) \oplus (\tilde{B}_{dR} \otimes V) \longrightarrow 0$$

inducing a long exact sequence

$$0 \longrightarrow H^0(K, V) \longrightarrow D_{N=0} \longrightarrow D_{N=0} \oplus t_V \longrightarrow H_f^1(K, V) \longrightarrow 0. \tag{S_f}$$

1.3. $H_{st}^1(K, V)$

Let

$$B'_{st} = \{(x, y) \in (B_{st})^2 \mid p\varphi x - x = Ny\}.$$

If $z \in B_{st}$, then $(Nz, \varphi z - z) \in B'_{st}$. We denote $\iota : B_{st} \rightarrow B'_{st} \oplus \tilde{B}_{dR}$ the map $z \mapsto ((Nz, \varphi z - z), \tilde{z})$.

Lemma 1. *The sequence*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{st} \xrightarrow{\iota} B'_{st} \oplus \tilde{B}_{dR} \longrightarrow 0 \tag{5}$$

is exact.

Proof. It is clear that $\ker(\iota) = B_{st}^{N=0, \varphi=1} \cap B_{dR}^+ = \mathbb{Q}_p$. We only need to show that ι is surjective. Let $((x, y), w) \in B'_{st} \oplus \tilde{B}_{dR}$. By surjectivity of $N : B_{st} \rightarrow B_{st}$, there is a $z_1 \in B_{st}$ such that $Nz_1 = x$. We have $N(y - (\varphi z_1 - z_1)) = p\varphi x - x - N(\varphi z_1 - z_1) = 0$, i.e. $y - (\varphi z_1 - z_1) \in B_{cris}$. By surjectivity of $\varphi - 1 : B_{cris} \rightarrow B_{cris}$, there is a $z_2 \in B_{cris}$ such that $\varphi z_2 - z_2 = y - (\varphi z_1 - z_1)$. By surjectivity of $B_e \rightarrow \tilde{B}_{dR}$, there is a $z_3 \in B_e$ such that $\tilde{z}_3 = w - (\tilde{z}_1 + \tilde{z}_2)$. Let $z = z_1 + z_2 + z_3 \in B_{st}$, then we have $\iota(z) = ((x, y), w)$. \square

Tensoring (5) with V , we get a short exact sequence

$$0 \longrightarrow V \longrightarrow B_{st} \otimes V \longrightarrow (B'_{st} \otimes V) \oplus (\tilde{B}_{dR} \otimes V) \longrightarrow 0$$

inducing a long exact sequence

$$0 \longrightarrow H^0(K, V) \longrightarrow D \longrightarrow D' \oplus t_V \longrightarrow H_{st}^1(K, V) \longrightarrow 0 \tag{S_{st}}$$

where $D' = H^0(K, B'_{st})$.

Moreover D' can be easily computed from D :

Proposition 2. *Denote $x \mapsto \bar{x}$ the projection of D onto D/ND and consider the maps*

$$\begin{aligned} \iota_0 : D_{N=0} &\longrightarrow D \oplus D_{N=0}, & w &\mapsto (w, -\varphi w + w), \\ \iota_1 : D \oplus D_{N=0} &\longrightarrow D \oplus D, & (u, v) &\mapsto (Nu, \varphi u - u + v), \\ \iota_2 : D' &\longrightarrow D/ND, & (x, y) &\mapsto \bar{x}. \end{aligned}$$

The image of ι_1 is contained in D' , the image of ι_2 is contained in $(D/ND)_{\varphi=p-1}$ and the sequence

$$0 \longrightarrow D_{N=0} \xrightarrow{\iota_0} D \oplus D_{N=0} \xrightarrow{\iota_1} D' \xrightarrow{\iota_2} (D/ND)_{\varphi=p-1} \longrightarrow 0$$

is exact.

Proof. The inclusions

$$\text{Image}(\iota_1) \subset D' \quad \text{and} \quad \text{Image}(\iota_2) \subset (D/ND)_{\varphi=p-1}$$

are obvious. We have:

$$D' = \{(x, y) \in D^2 \mid p\varphi x - x = Ny\}.$$

If $x \in D$ lifts $s \in (D/ND)_{\varphi=p-1}$, then there exists $y \in D$ such that $Ny = p\varphi x - x$ and (x, y) is in D' and such that $\iota_2(x, y) = s$, hence ι_2 is onto.

If $(u, v) \in D \oplus D_{N=0}$, we have $\iota_2(\iota_1(u, v)) = \iota_2(Nu, \varphi u - u + v) = 0$. Conversely, if $(x, y) \in D'$ lies in the kernel of ι_2 , it means there exists $u \in D$ such that $Nu = x$. Hence $(x, y) - \iota_1(u, 0)$ is an element of D' of the form $(0, v)$ and $Nv = 0$. Hence $(x, y) = \iota_1(u, v)$ and the image of ι_1 is the kernel of ι_2 .

If $w \in D_{N=0}$, then $\iota_1(\iota_0(w)) = \iota_1(w, -\varphi w + w) = (Nw, \varphi w - w - \varphi w + w) = 0$. Conversely, if (u, v) lies in the kernel of ι_1 , we have $Nu = 0$ and $v = -\varphi u + u$, hence $(u, v) = \iota_0(u)$.

The map ι_0 is obviously injective and it concludes the proof. \square

The following result is now obvious:

Proposition 3. The \mathbb{Q}_p -vector spaces $H_f^1(K, V)/H_e^1(K, V)$ and $H_{st}^1(K, V)/H_e^1(K, V)$ are finite dimensional. We have:

$$\dim_{\mathbb{Q}_p} H_f^1(K, V)/H_e^1(K, V) = \dim_{\mathbb{Q}_p} D_{N=0, \varphi=1}$$

and

$$\dim_{\mathbb{Q}_p} H_{st}^1(K, V)/H_f^1(K, V) = \dim_{\mathbb{Q}_p} (D/ND)_{\varphi=p-1}.$$

2. The case of a finite extension of \mathbb{Q}_p

We assume now that K is a finite extension of \mathbb{Q}_p . Recall that a p -adic Galois representation V of G_K is *potentially semi-stable* if there is a finite extension L of K contained in \bar{K} such that, if L_0 is the fraction field of the ring of Witt vectors with coefficients in the residue field of L :

$$\dim_{\mathbb{Q}_p} V = \dim_{L_0} H^0(L, B_{st} \otimes V).$$

In this case, we can use Proposition 3 to compute the dimension of $H_g^1(K, V)/H_f^1(K, V)$ and get Hyodo's celebrated result (cf. [6]):

Main Theorem. For a potentially semi-stable representation V ,

$$H_g^1(K, V) = H_{st}^1(K, V). \tag{*}$$

The original proof of Hyodo, never published, used decomposition of iso-crystals and unramified representations. This result has been extended by Laurent Berger [1] to the general case (Berger proves that any de Rham representation is potentially semi-stable), but his proof is much more involved.

2.1. Reduction to the semi-stable case

We consider the commutative diagram

$$\begin{array}{ccccc} H^1(K, V) & \xrightarrow{\alpha_K} & H^1(K, B_{st} \otimes V) & \xrightarrow{\beta_K} & H^1(K, B_{dR} \otimes V) \\ \downarrow \text{Res} & & \downarrow \text{Res} & & \downarrow \text{Res} \\ H^1(L, V) & \xrightarrow{\alpha_L} & H^1(L, B_{st} \otimes V) & \xrightarrow{\beta_L} & H^1(L, B_{dR} \otimes V) \end{array}$$

where L is a finite extension of K . The vertical arrows are injective by the relation $\text{Cor} \circ \text{Res} = [L : K]$. The above diagram shows that the injectivity of $\beta_L|_{\text{Im}(\alpha_L)}$ implies the injectivity of $\beta_K|_{\text{Im}(\alpha_K)}$.

By definition of $H_{st}^1(K, V)$ and $H_g^1(K, V)$, we have the following commutative diagram, where the two horizontal sequences are exact:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_{st}^1(K, V) & \longrightarrow & H^1(K, V) & \xrightarrow{\alpha_K} & \text{Im}(\alpha_K) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \beta_K|_{\text{Im}(\alpha_K)} \\
 0 & \longrightarrow & H_g^1(K, V) & \longrightarrow & H^1(K, V) & \longrightarrow & H^1(K, B_{dR} \otimes V)
 \end{array}$$

By the Snake Lemma, we know that $H_{st}^1(K, V) = H_g^1(K, V)$ is equivalent to the injectivity of $\beta_K|_{\text{Im}(\alpha_K)}$. So (*) for K is equivalent to (*) for L .

2.2. Computation of $\dim H_g^1(K, V)/H_f^1(K, V)$

Now assume V is semi-stable. Let $V^*(1)$ be the dual representation twisted by the Tate module of the multiplicative group. Recall the following result of Bloch and Kato [2, propo. 3.8]:

Lemma 4. *The usual perfect pairing of class field theory (given by the cup-product)*

$$H^1(K, V) \times H^1(K, V^*(1)) \longrightarrow H^2(K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p,$$

is such that

- (1) $H_e^1(K, V)$ and $H_g^1(K, V^*(1))$ are the exact annihilators of each other,
- (2) $H_g^1(K, V)$ and $H_e^1(K, V^*(1))$ are the exact annihilators of each other,
- (3) $H_f^1(K, V)$ and $H_f^1(K, V^*(1))$ are the exact annihilators of each other.

By the above Lemma, then

$$\dim_{\mathbb{Q}_p} H_g^1(K, V)/H_f^1(K, V) = \dim_{\mathbb{Q}_p} H_f^1(K, V^*(1))/H_e^1(K, V^*(1)).$$

By Proposition 3, the latter one is equal to

$$\dim_{\mathbb{Q}_p} D_{st}(V^*(1))_{N=0, \varphi=1} = \dim_{\mathbb{Q}_p} D_{st}(V^*)_{N=0, \varphi=p^{-1}}.$$

By duality, this is equal to

$$\dim_{\mathbb{Q}_p} ((D/ND)^*)^{\varphi=p^{-1}} = \dim_{\mathbb{Q}_p} (D/ND)^{\varphi=p^{-1}},$$

which is equal to $\dim_{\mathbb{Q}_p} H_{st}^1(K, V)/H_f^1(K, V)$ by using Proposition 3 again. This concludes the proof of the Main Theorem. □

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