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Partial differential equations/Numerical analysis

Optimal decay rates for the stabilization of a string network

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ARTICLE INFO

Article history:

Received 11 January 2014

Accepted after revision 31 March 2014

Available online 26 April 2014

Presented by Gilles Lebeau

ABSTRACT

We study the decay of the energy for a degenerate network of strings, and obtain optimal decay rates when the lengths are all equal. We also define a classical space semi-discretization and compare the results with the exact method introduced by Ammari and Jellouli (Appl. Math. 52 (4) (2007) 327–343; Bull. Belg. Math. Soc. Simon Stevin 4 (2010) 717–735).

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R É S U M É

On étudie la décroissance de l'énergie pour un réseau de cordes dégénéré, et on obtient des taux de décroissance optimaux lorsque les longueurs sont égales. On définit aussi une semi-discrétisation classique et on compare les résultats avec ceux de la méthode exacte introduite par Ammari et Jellouli (Appl. Math. 52 (4) (2007) 327–343; Bull. Belg. Math. Soc. Simon Stevin 4 (2010) 717–735).

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1. Introduction

We recall that the dissipation condition $\partial_x u(t, 0) = \alpha \partial_t u(t, 0)$, at the origin of a vibrating elastic string fixed at its end point, stabilizes this string. More precisely, if $E(t)$ denotes the energy of the solution of the wave equation $\partial_t^2 u(t, x) - \partial_x^2 u(t, x) = 0$ on $(0, \ell)$ subject to the initial condition $u(0, \cdot) = a$ and $\partial_t u(0, \cdot) = b$, then $E(t) \leq Ce^{-\gamma_\alpha t}$ where $\gamma_\alpha = \frac{1}{\ell} \log \left| \frac{1+\alpha}{1-\alpha} \right|$. Moreover, we remark that if $\alpha = 1$, then $E(t) = 0, \forall t \geq 2\ell$. Thus, the value 1 is the best choice of α that makes the system is an equilibrium state. The situation is completely different in the case when we consider a network of strings.

The problem of stabilization of nondegenerate network of strings was studied by K. Ammari, M. Jellouli and M. Khenissi in [5], [1] and [2]. When the strings are coupled at a common end in a star-shaped configuration, it is proven in [1] that the solutions are not exponentially stable in the energy space, in the nondegenerate case. In the particular case of two strings, which is equivalent to the pointwise stabilization of one string, such a problem has been studied in [4]; exponential stabilization is obtained if and only if the lengths satisfy $\frac{\ell_1}{\ell_1 + \ell_2} = \frac{p}{q}$, with p and q odd numbers, and the best decay rate, when fixing the total length $\ell_1 + \ell_2$, is obtained when the lengths are equal ($\ell_1 = \ell_2 = \ell$), with $\gamma = \frac{\ln(3)}{\ell}$ as the best decay

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rate. Such a problem has been also considered in [7], with different boundary conditions; in the degenerate case, the energy limit was identified and it was proven that the solution decays exponentially toward that limit.

We consider here the case of a *degenerate* network of vibrating elastic strings when the pointwise feedback acts in the root of the tree (tree-shaped network). Note that in the *nondegenerate* case, it is proven in [5] that the solutions are not exponentially stable in the energy space. We calculate the limit energy $E_\infty := \lim_{t \rightarrow +\infty} E(t)$ and we show that the decrease from $E(t)$ to E_∞ is exponential, giving the best decay rate, when the lengths are equal. Finally, we give numerical results in Section 3, which confirm the theoretical results.

2. E_∞ and best decay rate

Let $N \geq 3$. We consider the initial data $((a_j)_{1 \leq j \leq N}, (b_j)_{1 \leq j \leq N}) \in \mathcal{H} := \prod_{j=1}^N H^2(0, \ell_j) \times \prod_{j=1}^N H^1(0, \ell_j)$, satisfying the compatibility conditions

$$a'_1(0) = \alpha b_1(0), \quad a'_1(\ell_1) = \sum_{j=2}^N a'_j(0) \quad \text{and} \quad a_j(\ell_j) = 0, \quad a_1(\ell_1) = a_j(0), \quad j = 2, \dots, N, \tag{1}$$

and the system of partial differential equations

$$(S) : \begin{cases} \partial_t^2 u_j(t, x) - \partial_x^2 u_j(t, x) = 0, & t > 0, x \in (0, \ell_j), j = 1, \dots, N, \\ u_j(t, 0) = u_1(t, \ell_1) \quad \text{and} \quad u_j(t, \ell_j) = 0, & t \geq 0, j = 2, \dots, N, \\ \partial_x u_1(t, 0) = \alpha \partial_t u_1(t, 0), & t \geq 0 (\alpha > 0), \\ \partial_x u_1(t, \ell_1) = \sum_{j=2}^N \partial_x u_j(t, 0), & t \geq 0, \\ u_j(0, x) = a_j(x), \quad \partial_t u_j(0, x) = b_j(x), & x \in [0, \ell_j], j = 1, \dots, N. \end{cases}$$

We define the energy of the solution of (S) by

$$E(t) = \frac{1}{2} \sum_{j=1}^N \|\partial_t u_j(t)\|_{L^2(0, \ell_j)}^2 + \frac{1}{2} \sum_{j=1}^N \|\partial_x u_j(t)\|_{L^2(0, \ell_j)}^2 \tag{2}$$

and we prove the following theorems in the case where $\ell_j = \ell, j = 1, \dots, N$ (we denote by $\|\cdot\|$ the norm of $L^2(0, \ell)$):

Theorem 2.1. *If $((a_j)_{1 \leq j \leq N}, (b_j)_{1 \leq j \leq N}) \in \mathcal{H}$ verify (1), then the energy limit is given by*

$$E_\infty = \frac{1}{2(N-1)} \sum_{j=2}^N \sum_{k=j+1}^N (\|a'_k - a'_j\|^2 + \|b_k - b_j\|^2). \tag{3}$$

Now, we denote by $\alpha_0 = \frac{2\sqrt{N-1}}{N}, \Delta = \frac{4(\alpha^2 N^2 - 4N + 4)}{(\alpha + 1)^2}$ and $\lambda = -\frac{2(N-2)}{\alpha + 1} + \sqrt{\Delta}$.¹

Theorem 2.2. *There is a constant $C = C_N(\alpha) > 0$ such that for any initial data $((a_j)_{1 \leq j \leq N}, (b_j)_{1 \leq j \leq N}) \in \mathcal{H}$ satisfying (1) the following inequality holds:*

$$0 \leq E(t) - E_\infty \leq C \left(\|a'_1\|^2 + \|b_1\|^2 + \left\| \sum_{j=2}^N a'_j \right\|^2 + \left\| \sum_{j=2}^N b_j \right\|^2 \right) \frac{e^{-\gamma t}}{|\Delta|}, \quad \text{if } \alpha \neq \alpha_0, \tag{4}$$

$$0 \leq E(t) - E_\infty \leq C \left(\|a'_1\|^2 + \|b_1\|^2 + \left\| \sum_{j=2}^N a'_j \right\|^2 + \left\| \sum_{j=2}^N b_j \right\|^2 \right) t^2 e^{-\gamma_0 t}, \quad \text{if } \alpha = \alpha_0, \tag{5}$$

where $\gamma = \frac{1}{\ell} \log \frac{1}{|\lambda|} > 0$ and $\gamma_0 = \frac{1}{\ell} \log \frac{N+2\sqrt{N-1}}{N-2}$. The best decay rate is achieved in the sense that for all $\alpha > 0$, there exists initial data such that (4) and (5) become an equivalence:

$$E(t) - E_\infty \geq C e^{-\gamma t}, \quad \text{or, for } \alpha = \alpha_0, \quad E(t) - E_\infty \geq C t^2 e^{-\gamma_0 t}, \quad \text{with } C > 0,$$

and $\forall \alpha \neq \alpha_0$, we have $\gamma < \gamma_0$.

Remark 1. The case $N = 2$ (with ℓ_1, ℓ_2 not necessarily of same size) can be recasted into the case $N = 1$ (with $\ell = \ell_1 + \ell_2$).

¹ $\sqrt{\Delta} = i\sqrt{-\Delta}$ if $\Delta < 0$. Note that, $\Delta < 0$ (resp. $\Delta = 0$) if and only if $\alpha \in]0, \alpha_0[$ (resp. $\alpha = \alpha_0$).

The proofs of these theorems are based on the operator of type τ developed in [2] and [3], and will be detailed in [6]. In fact we recall that

$$E(t) = E(0) - \alpha \int_0^t |\partial_t u_1(s, 0)|^2 ds \tag{6}$$

and that for all $t \geq 0$, $\partial_t u_1(t, 0) = (\mathcal{P} \circ \tilde{\mathcal{P}}^{-1})F(t) + \frac{1}{\alpha+1}(\mathcal{L}_{\beta, \ell, 1}^+ a'_1 + \mathcal{L}_{\beta, \ell, 1}^- b_1)$, $\beta = \frac{\alpha-1}{\alpha+1}$.

In the first step, we evaluate the characteristic elements of $\tilde{\mathcal{P}}^{-1}$ and we prove that $\tilde{\mathcal{P}}^{-1} \equiv (n, \delta)$, where $n : n(j) = 2(j-1)\ell$, $j \geq 1$ and, in the case $\alpha \neq \alpha_0$:

$$\delta : \delta(j) = a(\beta)\lambda^{j-2} + b(\beta)\mu^{j-2}, \quad j \geq 2$$

with $\mu = -\frac{2(N-2)}{\alpha+1} - \sqrt{\Delta}$, $a(\beta) = \frac{1}{N} \frac{(1-\lambda)(\beta-\lambda)}{\lambda-\mu}$ and $b(\beta) = -\frac{1}{N} \frac{(1-\mu)(\beta-\mu)}{\lambda-\mu}$. In the case $\alpha = \alpha_0$,

$$\delta(j) = \left[\delta(2) + \left(\frac{1}{\lambda_0} \delta(3) - \delta(2) \right) (j-2) \right] \lambda_0^{j-2}, \quad \forall j \geq 2.$$

In the second stage, we calculate $\int_0^{2n\ell} |\partial_t u_1(s, 0)|^2 ds$, what will allow us to find E_∞ :

$$\begin{aligned} E_\infty &= \frac{1}{2} \sum_{j=2}^N (\|a'_j\|^2 + \|b_j\|^2) - \frac{1}{4(N-1)} \left(\left\| \sum_{j=2}^N (a'_j - b_j) \right\|^2 + \left\| \sum_{j=2}^N (a'_j + b_j) \right\|^2 \right) \\ &= \frac{1}{2(N-1)} \sum_{j=2}^N \sum_{k=j+1}^N (\|(a'_k - a'_j)\|^2 + \|(b_k - b_j)\|^2). \end{aligned}$$

In the end, we estimate the difference $E(t = 2n\ell) - E_\infty$ and we show the inequalities (4) and (5).

3. Semi-discretization and numerical results

Let N_j , $j = 1, \dots, d$ be positive integers. We define $h = (h_1, \dots, h_d)$, with $h_j = \frac{\ell_j}{N_j}$, $j = 1, \dots, d$. We suppose that $\alpha > 0$ and we consider the following classical finite difference space semi-discretization of (S):

$$\left\{ \begin{aligned} &u''_{j,k}(t) - \frac{u_{j,k-1}(t) - 2u_{j,k}(t) + u_{j,k+1}(t)}{h_j^2} = 0, \quad k = 1, \dots, N_j - 1, \quad j = 1, \dots, d, \\ &\frac{u_{1,1}(t) - u_{1,0}(t)}{h_1} = \alpha u'_{1,0}(t), \\ &\frac{u_{1,N_1}(t) - u_{1,N_1-1}(t)}{h_1} = \sum_{j=2}^d \frac{u_{j,1}(t) - u_{j,0}(t)}{h_j}, \\ &u_{1,N_1}(t) = u_{j,0}(t) \quad \text{and} \quad u_{j,N_j}(t) = 0, \quad j = 2, \dots, d, \\ &u_{j,k}(0) = u_{j,k}^0, \quad u'_{j,k}(0) = u_{j,k}^1, \quad j = 1, \dots, d, \quad k = 0, \dots, N_j, \quad t \in (0, \infty), \end{aligned} \right. \tag{7}$$

where the initial conditions satisfy, for $s = 0, 1$, the compatibility conditions:

$$u_{1,N_1}^s = u_{k,0}^s = \frac{\sum_{j=2}^d \frac{u_{j,1}^s}{h_j} + \frac{u_{1,N_1-1}^s}{h_1}}{\sum_{j=1}^d \frac{1}{h_j}} \quad \text{and} \quad u_{k,N_k}^s = 0, \quad k = 2, \dots, d.$$

The solution of (7) can be written in the form $V(t) = \exp(tA)\phi$, where $V = (U, U')^t$,

$$U = (u_{1,0}, u_{1,1}, \dots, u_{1,N_1-1}, u_{2,1}, \dots, u_{2,N_2-1}, \dots, u_{d,1}, \dots, u_{d,N_d-1}) \in \mathbb{R}^M$$

and $\phi \in \mathbb{R}^{2M}$. The semi-discrete energy is defined by

$$E_{h,\phi}(t) = \frac{1}{2} \sum_{j=1}^d h_j \left(\sum_{k=1}^{N_j-1} |u'_{j,k}(t)|^2 + \sum_{k=0}^{N_j-1} \left| \frac{u_{j,k+1}(t) - u_{j,k}(t)}{h_j} \right|^2 \right), \tag{8}$$

and is decreasing as it satisfies $E'_{h,\phi}(t) = -\alpha |u'_{1,0}(t)|^2$. The following proposition identifies the limit of the discrete energy, which may not be zero, as in the continuous case.

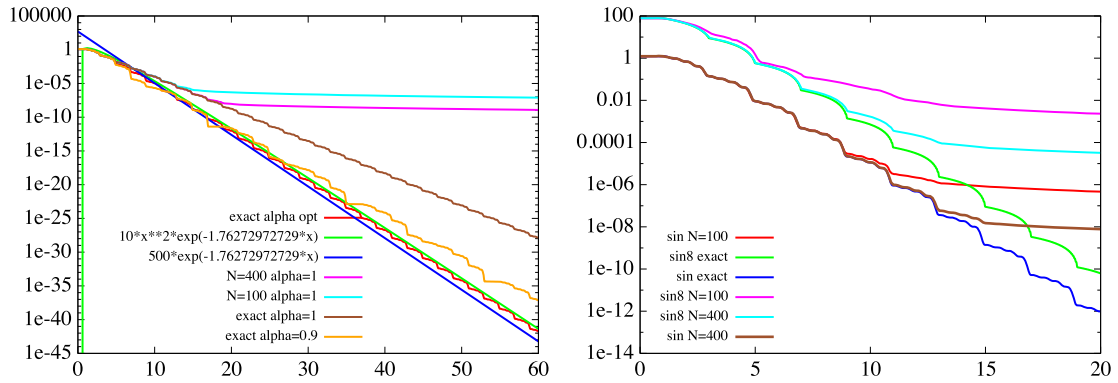


Fig. 1. $E(t) - E_\infty$ vs time (exact and semi-discrete case) for $\alpha \in \{0.9, 1, \alpha_{opt}\}$ (left) and different initial conditions (right).

Proposition 3.1. For $\phi \in \mathbb{R}^{2M}$, we can write $\phi = \phi_1 + \phi_2$, where ϕ_1 belongs to the space Λ_1 of eigenvectors relative to the eigenvalues λ of A , with $\Re(\lambda) = 0$ and ϕ_2 belongs to the space Λ_2 of generalized eigenvectors relative to the eigenvalues λ of A , with $\Re(\lambda) < 0$. We then have:

$$E_{h,\phi,\infty} := \lim_{t \rightarrow \infty} E_{h,\phi}(t) = E_{h,\phi_1}(0), \tag{9}$$

and this value does not depend on $\alpha > 0$.

Sketch of the proof. We first prove that for $\phi_1 \in \Lambda_1 \subset \mathbb{C}^{2M}$, we have $\phi_{1,M+1} = 0$, that no eigenvalue of A can have positive real part and that generalized eigenvectors with eigenvalue of zero real part are necessarily eigenvectors, using the fact that the energy is decreasing and nonnegative. We then use the decomposition and estimates:

$$\begin{aligned} & |E_{h,\phi}(t) - E_{h,\phi_1}(t) - E_{h,\phi_2}(t)| \\ &= \left| \sum_{j=1}^d \sum_{k=1}^{N_j-1} h_j (w_{j,k}^1)' (w_{j,k}^2)' + \sum_{j=1}^d \sum_{k=0}^{N_j-1} \frac{1}{h_j} (w_{j,k+1}^1 - w_{j,k}^1) (w_{j,k+1}^2 - w_{j,k}^2) \right| \\ &\leq \sum_{j=1}^d h_j \sqrt{\sum_{k=1}^{N_j-1} |(w_{j,k}^1)'|^2} \sqrt{\sum_{k=1}^{N_j-1} |(w_{j,k}^2)'|^2} + \frac{1}{h_j} \sqrt{\sum_{k=0}^{N_j-1} |w_{j,k+1}^1 - w_{j,k}^1|^2} \sqrt{\sum_{k=0}^{N_j-1} |w_{j,k+1}^2 - w_{j,k}^2|^2} \\ &\leq 2d \sqrt{E_{h,\phi_1}(t)} \sqrt{E_{h,\phi_2}(t)}, \end{aligned}$$

where $w_{j,k}^1$ (resp. $w_{j,k}^2$) is the solution corresponding to the initial condition ϕ_1 (resp. ϕ_2). We know that $E_{h,\phi_1}(t) = E_{h,\phi_1}(0)$, thus we get (9) since $E_{h,\phi_2}(t)$, being a finite sum and product of terms tending toward zero, as $t \rightarrow +\infty$, also tends toward zero. This limit does not depend on α , since the only equation where α is present is:

$$\alpha \lambda \phi_{\lambda,M+1} = 0 = \frac{1}{h_1} (\phi_{\lambda,M+2} - \phi_{\lambda,M+1}), \quad \text{for each } \lambda; \Re(\lambda) = 0.$$

Numerical results. Finally we give some exact and semi-discrete results for the energy. We take $d = 3$, initial condition

$$u_j^0(x) = 0, \quad j = 1, 3, \quad u_2^0(x) = \sin^2\left(\frac{\pi x}{\ell_2}\right) \quad \text{and} \quad u_j^1(x) = 0, \quad j = 1, 2, 3.$$

$\ell_j = 1$ and $N_j = N, j = 1, 2, 3$. We see in Fig. 1 (left) the time evolution of $E(t) - E_\infty$ for $\alpha = 1$ (exact, $N = 100, 400$) and for $\alpha = \alpha_{opt}, 0.9$ (exact). In Fig. 1 (right), we plot the time evolution of $E(t) - E_\infty$ (exact, $N = 100, 400$) taking as before $u_2^0(x) = \sin^2(\pi x)$ (sin) but also $u_2^0(x) = \sin^2(8\pi x)$ (sin8), in the case $\alpha = \alpha_{opt}$. The results conform with Theorem 2.2: taking $\alpha = \alpha_{opt}$ leads to the best growth rate with an asymptotic behavior of the form $Ct^2 \exp(-\gamma t)$. We also note that the semi-discrete scheme is able to reproduce the behavior of the exact scheme until the discretization error dominates, which is higher by taking a higher mode (Fig. 1, right). The semi-discrete energy seems to converge to a discretization-dependent limit, which is coherent with Proposition 3.1.

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