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Differential geometry

A note on magnetic curves on $\mathbb{S}^{2n+1}$

Une note sur les courbes magnétiques de $\mathbb{S}^{2n+1}$

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ABSTRACT

We prove that a normal magnetic curve on the Sasakian sphere $\mathbb{S}^{2n+1}$ lies on a totally geodesic sphere $\mathbb{S}^3$, and that the Sasakian structure on $\mathbb{S}^3$ is that induced from $\mathbb{S}^{2n+1}$.

RÉSUMÉ

Nous montrons qu'une courbe magnétique normale sur la sphère sasakienne $\mathbb{S}^{2n+1}$ se trouve sur une sphère totalement géodésique $\mathbb{S}^3$, et que la structure sasakienne sur $\mathbb{S}^3$ est celle qui est induite de $\mathbb{S}^{2n+1}$.

1. Introduction

In a previous paper [4], we studied magnetic curves in Sasakian and cosymplectic manifolds. More precisely, we showed that, essentially, the study of magnetic curves in Sasakian space forms of arbitrary dimension $\mathcal{M}^{2n+1}(c)$ reduces to their study in dimension 3. In particular, we proved some results on the reduction of the codimension. When $\mathcal{M}^{2n+1}(c)$ is the odd dimensional sphere, we proved the following result. *If γ is a normal magnetic curve on the (Sasakian) sphere $\mathbb{S}^{2n+1}$, then γ lies on the totally geodesic sphere $\mathbb{S}^3$*. Magnetic curves in 3-dimensional Sasakian manifolds were studied, for example, in [2,3]. Yet, in [4] we omitted the proof that the Sasakian structure on $\mathbb{S}^3$ is that induced from the ambient $\mathbb{S}^{2n+1}$. We only proved that ξ is tangent to $\mathbb{S}^3$ along the curve γ, since ξ belongs to the 2-plane spanned by the tangent $T = \dot{\gamma}$ and the second normal ν_2. In this note, we clarify these aspects. More precisely, we prove the following theorem.

Theorem. *Let γ be a normal magnetic curve on the standard Sasakian sphere $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$, corresponding to the contact magnetic field F_q. Then γ is a normal magnetic curve on a 3-dimensional sphere $\mathbb{S}^3(1)$, embedded as a Sasakian totally geodesic submanifold in $\mathbb{S}^{2n+1}(1)$.*

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2. Preliminaries

One of the topics situated at the interplay between differential geometry and physics is represented by the study of magnetic fields on Riemannian manifolds and of their corresponding magnetic curves. A closed 2-form F on a (complete) Riemannian manifold (M,g) is called a *magnetic field*. The *Lorentz force* on (M,g,F) is the skew symmetric $(1,1)$-type tensor field Φ on M satisfying $g(\Phi(X),Y)=F(X,Y)$, for any $X,Y\in\mathfrak{X}(M)$. Finally, a *trajectory* generated by the magnetic field F is defined as a smooth curve γ on M fulfilling the *Lorentz equation* $\nabla_{\dot\gamma}\dot\gamma=\Phi(\dot\gamma)$, where ∇ denotes the Levi Civita connection of g. For more details, see, e.g., [1].

A manifold M^{2n+1} is said to have an *almost contact metric structure*, if there exist a 1-form η, a vector field ξ, a $(1,1)$-type tensor field φ and a Riemannian metric g such that:

$$\eta(\xi)=1,\quad \varphi^2=-\mathrm{I}+\eta\otimes\xi,\quad \varphi\xi=0,\quad \eta\circ\varphi=0,$$
$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),\quad \forall X,Y\in\mathfrak{X}\big(M^{2n+1}\big).$$

An almost contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ is called *Sasakian* if

$$(\nabla_X\varphi)Y=-g(X,Y)\xi+\eta(Y)X,$$

for any $X,Y\in\mathfrak{X}(M^{2n+1})$, where ∇ is the Levi Civita connection on M^{2n+1}. As a consequence, we have $\nabla_X\xi=\varphi X$, $\forall X\in\mathfrak{X}(M^{2n+1})$. Note that we use the sign convention from [5].

Let $\mathbb{S}^{2n+1}$ be the unit sphere endowed with the canonical Sasakian structure (φ,ξ,η,g) induced from the Kähler structure of $\mathbb{C}^{n+1}$. More precisely, if J denotes the canonical complex structure on $\mathbb{C}^{n+1}\equiv\mathbb{R}^{2(n+1)}$, and a point $p\in\mathbb{S}^{2n+1}$ is identified with its position vector, we have

$$\xi=Jp,\quad \eta(X)=-\langle JX,p\rangle,\quad \varphi X=JX+\eta(X)p,$$

for all X tangent to $\mathbb{S}^{2n+1}$. The metric g is that induced from the Euclidean product $\langle\,,\,\rangle$ of $\mathbb{R}^{2(n+1)}$.

A smooth curve $\gamma:I\subset\mathbb{R}\longrightarrow\mathbb{S}^{2n+1}$ parameterized by arc length is called a *normal magnetic curve* on $\mathbb{S}^{2n+1}$ if its velocity satisfies the Lorentz equation

$$\nabla_{\dot\gamma}\dot\gamma=q\varphi\dot\gamma, \tag{1}$$

where $q\neq0$ is a real constant called the *strength*.

Since $\mathbb{S}^{2n+1}$ is embedded in the Euclidean space $\mathbb{R}^{2(n+1)}$, one may write the Lorentz equation (1) in the ambient space as follows:

$$\ddot\gamma+\gamma=q\big(J\dot\gamma+\cos\theta(s)\gamma\big), \tag{2}$$

where θ is the angle function between the unit tangent $\dot\gamma$ and the characteristic vector field ξ in $\gamma(s)$. In fact, it was proved that the angle θ is constant. See for example [4].

Subsequently, we have to study curves $\gamma:I\longrightarrow\mathbb{C}^{n+1}$ satisfying

$$\ddot\gamma+ia\dot\gamma+b\gamma=0, \tag{3}$$

where a,b are real constants. In our case $a=-q$ and $b=1-q\cos\theta$.

3. Reduction of the codimension: proof of the theorem

Let $z_1,\dots,z_{n+1}$ be global coordinates on $\mathbb{C}^{n+1}$. For any $k\in\{1,\dots,n+1\}$ write $z=z_k$. Then, Eq. (3) reduces to

$$\ddot z+ia\dot z+bz=0.$$

In order to prove our statement, we have to solve first this equation, which is equivalent to

$$\ddot w+\left(\frac{a^2}{4}+b\right)w=0, \tag{4}$$

where we put

$$w=\left(\sin\frac{as}{2}-\mathrm{i}\cos\frac{as}{2}\right)z. \tag{5}$$

As $\frac{a^2}{4}+b>0$, when $\sin\theta\neq0$ we immediately obtain the solution:

$$w(s)=\alpha\cos(cs)+\beta\sin(cs), \tag{6}$$

where $c=\sqrt{\frac{a^2}{4}+b}$ and α,β are arbitrary complex constants.

Going back to Eq. (3), we are able to write the general solution

$$z_k(s) = \left(\cos\frac{qs}{2} + \mathrm{i}\sin\frac{qs}{2}\right)\big(\alpha_k\cos(cs) + \beta_k\sin(cs)\big),$$

where $\alpha_k,\ \beta_k \in \mathbb{C}$, $k = 1, \dots, n+1$. Hence, the curve γ in $\mathbb{R}^{2(n+1)}$ may be written as

$$\gamma(s) = \cos\frac{qs}{2}\cos(cs)V_1 + \sin\frac{qs}{2}\cos(cs)V_2 + \cos\frac{qs}{2}\sin(cs)V_3 + \sin\frac{qs}{2}\sin(cs)V_4,$$

where $V_1, \dots, V_4$ are constant vectors in $\mathbb{R}^{2(n+1)}$ of the form

$$V_1 = (v_1, v_2), \qquad V_2 = (-v_2, v_1), \qquad V_3 = (v_3, v_4), \qquad V_4 = (-v_4, v_3),$$

with $v_1, \dots, v_4 \in \mathbb{R}^{n+1}$. Since $\gamma(s)$ is unitary, we obtain

$$\cos^2(cs)\big(|v_1|^2 + |v_2|^2\big) + \sin^2(cs)\big(|v_3|^2 + |v_4|^2\big) + \sin(cs)\cos(cs)\big(\langle v_1, v_3\rangle + \langle v_2, v_4\rangle\big) = 1, \quad \forall s.$$

Hence

$$|v_1|^2 + |v_2|^2 = |v_3|^2 + |v_4|^2 = 1, \qquad \langle v_1, v_3\rangle + \langle v_2, v_4\rangle = 0,$$

i.e. V_1, V_2, V_3, V_4 are unitary and $V_1 \perp V_3$, $V_2 \perp V_4$.

The fact that γ is parameterized by arc length implies that

$$\langle V_1, V_4\rangle = \frac{1 - 2q\cos\theta}{c} = -\langle V_2, V_3\rangle.$$

If V_1 and V_4 are collinear, then V_2 and V_3 are collinear too, and hence

$$\gamma(s) = \cos\left(\frac{q}{2} - \epsilon c\right)sV_1 + \sin\left(\frac{q}{2} - \epsilon c\right)sV_2, \quad \epsilon = \pm 1,$$

which is a circle of radius 1 in a 2-plane in $\mathbb{R}^{2(n+1)}$ invariant by J. The tangent vector $\dot\gamma(s)$ coincides with $\pm\xi_{|_{\gamma(s)}}$, hence $\sin\theta = 0$.

If V_1 and V_4 are linearly independent, then one may consider the 4-dimensional real vector space $\mathcal{V}_4 = \mathrm{span}\{V_1, \dots, V_4\}$, which is a J-invariant subspace in $\mathbb{R}^{2(n+1)}$. Since $V_1, \dots, V_4$ are constant vectors, the space $\mathcal{V}_4$ is independent of s.

We have proved that γ lies on the unit 3-sphere $\mathbb{S}^3$ as hypersurface in $\mathcal{V}_4$. But the standard Sasakian structures on the two spheres are obtained from the same complex structure J, since $\mathcal{V}_4$ is J invariant. It follows that $\mathbb{S}^3$ carries the induced Sasakian structure of $\mathbb{S}^{2n+1}$.

When $\sin\theta = 0$, then $\dot\gamma$ is collinear to ξ. From the Lorentz equation we have that γ is a geodesic in $\mathbb{S}^{2n+1}$ as integral curve of ξ.

Hence the theorem is proved.

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