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Differential geometry

A characterization of balanced manifolds

*Une caractérisation des variétés semi-kählériennes*

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ABSTRACT

A Hermitian metric on a complex manifold is Kähler if and only if it approximates the Euclidean metric to order 2 at each point, in a suitable coordinate system. We prove here an analogous characterization of balanced metrics, namely, a Hermitian metric is balanced if and only if its fundamental form ω has closed trace and $\omega_{i,j}(z)$ does not contain linear terms involving $z_i, z_j, \bar{z}_i, \bar{z}_j$, for each point, in a suitable coordinate system.

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R É S U M É

Une métrique hermitienne de forme fondamentale ω sur une variété complexe M est kählérienne si et seulement s'il existe un système de coordonnées z sur un voisinage de chaque point de M , tel que la composante linéaire de $\omega_{i,j}(z)$ s'annule. On montre ici un critère de semi-kählérianité, à savoir qu'une métrique hermitienne de forme ω sur M est semi-kählérienne si et seulement s'il existe un système de coordonnées z sur un voisinage de chaque point de M , tel que la part linéaire de $\omega_{i,j}(z)$ ne contienne pas $z_i, z_j, \bar{z}_i, \bar{z}_j$, et que la trace de ω soit fermée.

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1. Introduction

The general context of this note is the wide collection of works originated by the paper of M.L. Michelson in 1983 [5], where balanced manifolds were introduced and studied; we also refer to our joint papers in collaboration with G. Bassanelli on this subject (see, e.g., [1,2]).

The point of view of Michelson is the following: “the condition of being balanced is, in a strong sense, dual to that of being Kähler”, because she starts from a very general problem, namely, how to choose a good Hermitian metric on a complex manifold.

By contrast, our philosophy is to consider Kähler and balanced manifolds from the point of view of p -Kähler manifolds (see Section 2): Kähler manifolds correspond to the case $p = 1$, while balanced manifolds correspond to the case $p = n - 1$, where n is the dimension of the manifold.

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The approaches are quite different: in the first case, one studies Hermitian metrics, in the other one, only positive differential forms.

Starting from a Hermitian metric h , we can consider both the associated canonical Hermitian connection and the Kähler form of the metric. In the first case, we can look at the torsion tensor T_h of h , and at the torsion 1-form τ_h of h : h is Kähler if and only if $T_h = 0$, h is balanced if and only if $\tau_h = 0$.

Alternatively, if ω_h is the Kähler form of h , and d_h^* is the formal adjoint of d with respect to the metric h , one gets that h is Kähler if and only if $d\omega_h = 0$, h is balanced if and only if $d_h^*\omega_h = 0$, that is, if and only if $d(\omega_h)^{n-1} = 0$.

Notice that, at an elementary level, the notion of Kähler metric is introduced in a different way: since it is important, especially on compact manifolds, to ensure a link between Laplacians, one says that a Hermitian metric is Kähler if and only if it approximates the Euclidean metric to order 2 at each point [4, p. 106]. In this paper, we investigate to which extent balanced metrics are dual to Kähler metrics from this point of view, which we feel to be the easiest and the most basic one to look at. The case $n = 2$ allows us to compare our result with the classical one.

This note will pursue our past philosophy of considering $(n - 1)$ -Kähler forms (see Theorem 3.1). As a corollary (see Proposition 3.3) we give the characterization result of balanced metrics. The motivation is to stress that, also when we do not have a good Hermitian metric on the manifold, it is possible to handle and solve problems at the level of differential forms.

2. Notation and preliminary results

Let M be a complex manifold of dimension $n \geq 3$, let p be an integer, $1 \leq p \leq n - 1$, and let $\sigma_p = i^{p^2} 2^{-p}$.

Definition 2.1. M is a p -Kähler manifold if it has a closed transverse (i.e. strictly weakly positive) (p, p) -form Ω , which is called a p -Kähler form.

For $p = 1$, a transverse form is the fundamental form of a Hermitian metric, so that a 1-Kähler manifold is simply a Kähler manifold, and we can look at 1-Kähler metrics.

The case $p = n - 1$ was studied by Michelson in [5], where $(n - 1)$ -Kähler manifolds are called *balanced* manifolds.

For $p = n - 1$, we get a Hermitian metric too, because every transverse $(n - 1, n - 1)$ -form Ω is in fact given by $\Omega = \omega^{n-1}$, where ω is a transverse $(1, 1)$ -form (the proof uses a comparison between the eigenvalues of Ω_x and those of ω_x , see [5, p. 279]); we say that ω is associated with a balanced metric.

When $n = 2$, a balanced metric (manifold) is simply a Kähler metric (manifold).

3. Characterization of balanced manifolds

Theorem 3.1. Let M be a complex n -dimensional manifold, and let Ω be a real $(n - 1, n - 1)$ -form on M . Then Ω is an $(n - 1)$ -Kähler form if and only if, for every $p \in M$, there is a holomorphic coordinate system (w_1, \dots, w_n) centered at p such that

$$\Omega = \sigma_{n-1} \sum_{i,j=1}^n \Omega_{i,\bar{j}} \widehat{dw}_i \wedge \widehat{d\bar{w}}_j,$$

with

- (i) $\Omega_{i,\bar{j}}(0) = \delta_{i,j}$,
- (ii) $\Omega_{i,\bar{j}}(w)$ does not contain linear terms involving $w_i, w_j, \bar{w}_i, \bar{w}_j$,
- (iii) $d(\text{tr } \Omega_{i,\bar{j}})(0) = 0$.

Proof. One part of the proof is easy: in fact, when Ω satisfies (i) for every $p \in M$, then $\Omega|_p = \sigma_{n-1} \sum_{i=1}^n \widehat{dw}_i \wedge \widehat{d\bar{w}}_i > 0$. Moreover,

$$\partial\Omega|_p = 0 \iff \forall j, \sum_{i=1}^n (-1)^{i-1} (\partial_i \Omega_{i,\bar{j}})(0) = 0. \tag{1}$$

By condition (ii), $\Omega_{i,\bar{j}}$ does not contain linear terms involving w_i , so that $\partial\Omega|_p = 0$, and thus $d\Omega|_p = 0$, that is, Ω is closed.

Let now suppose that Ω is an $(n - 1)$ -Kähler form, let $p \in M$, and let (z_1, \dots, z_n) be a generic holomorphic coordinate system centered at p (that is, $z_j(p) = 0$). Here Ω is given by $\Omega = \sigma_{n-1} \sum_{i,j=1}^n \Omega_{i,\bar{j}} \widehat{dz}_i \wedge \widehat{d\bar{z}}_j$; since $\Omega > 0$, we can choose (z_1, \dots, z_n) such that $\Omega|_p$ is diagonalized, that is, $\Omega_{i,\bar{j}}(0) = \delta_{i,j}$. Thus

$$\Omega_{i,\bar{j}}(z) = \delta_{i,j} + \sum_{k=1}^n (A_{i,\bar{j},k} z_k + A_{i,\bar{j},\bar{k}} \bar{z}_k) + (\dots) \tag{2}$$

where (...) will always denote some terms of order at least two, with respect to the given coordinates.

Take another holomorphic coordinate system (w_1, \dots, w_n) centered at p , such that

$$z_k = w_k + \frac{1}{2} \sum_{r,s=1}^n b_{k,r,s} w_r w_s + (\dots). \tag{3}$$

We get: $\Omega = \sigma_{n-1} \sum_{i,j=1}^n \Omega'_{i,\bar{j}} \widehat{dw}_i \wedge \widehat{d\bar{w}}_j$, with

$$\Omega'_{i,\bar{j}}(w) = \delta_{i,j} + \sum_{k=1}^n (C_{i,\bar{j},k} w_k + C_{i,\bar{j},\bar{k}} \bar{w}_k) + (\dots). \tag{4}$$

We collect in the following lemma all relations among the coefficients A, b, C ; the proof is done by a routine computation, and is given at the end of the present proof.

Lemma 3.2.

- (i) $\forall i, j, k, b_{i,k,j} = b_{i,j,k}$ (symmetry).
- (ii) $\forall i, j, k, C_{i,\bar{j},\bar{k}} = \overline{C_{j,\bar{i},k}}$ (Ω is real).
- (iii) $\forall j, \sum_{i=1}^n (-1)^i A_{i,\bar{j},i} = 0, \sum_{i=1}^n (-1)^i C_{i,\bar{j},i} = 0$ (Ω is closed).
- (iv) $\forall i, j, k$ (with $i \neq j$), $C_{i,\bar{j},k} = A_{i,\bar{j},k} + (-1)^{i+j-1} b_{i,k,j}$.
- (v) $\forall i, k, C_{i,\bar{i},k} = A_{i,\bar{i},k} + \sum_{r \neq i} b_{r,k,r}$.
- (vi) $\forall i, k, b_{i,k,i} = A_{i,\bar{i},k} - C_{i,\bar{i},k} + \frac{1}{n-1} \sum_{r=1}^n (C_{r,\bar{r},k} - A_{r,\bar{r},k})$.
- (vii) $\forall i, j, k$ distinct indices, $(-1)^{i+j} (A_{i,\bar{j},k} - C_{i,\bar{j},k}) = (-1)^{i+k} (A_{i,\bar{k},j} - C_{i,\bar{k},j})$.
- (viii) $\forall i, j$ (with $i \neq j$), $C_{i,\bar{i},j} = A_{i,\bar{i},j} - (-1)^{i+j} A_{i,\bar{j},i} + \frac{1}{n-1} (\sum_{r=1}^n C_{r,\bar{r},j} - \sum_{r=1}^n A_{r,\bar{r},j})$.

Summing up, we can choose freely $\{C_{i,\bar{j},k}, C_{i,\bar{j},\bar{k}}\}$ (or $\{b_{i,j,k}\}$, which is the same, by the lemma), only respecting conditions (ii), (iii), (vii), (viii) in the lemma.

Notice also that:

$\Omega_{i,\bar{j}}(w)$ does not contain linear terms involving $w_i, w_j, \bar{w}_i, \bar{w}_j$ if and only if $C_{i,\bar{j},i} = C_{i,\bar{j},j} = C_{i,\bar{j},\bar{i}} = C_{i,\bar{j},\bar{j}} = 0$, and $d(\text{tr } \Omega_{i,\bar{j}})(0) = 0$ if and only if $\forall j, \sum_{i=1}^n C_{i,\bar{i},j} = 0$.

Thus we choose $\{C_{i,\bar{j},k}\}$, for distinct indices, such that (vii) holds; then we set $\forall i, j, C_{i,\bar{j},i} = C_{i,\bar{j},j} = 0$ (also when $i = j$), so that (iii) is guaranteed.

It remains only to choose $\{C_{i,\bar{i},j}\}$ when $i \neq j$; put

$$C_{i,\bar{i},j} = A_{i,\bar{i},j} - (-1)^{i+j} A_{i,\bar{j},i} - \frac{1}{n-1} \sum_{r=1}^n A_{r,\bar{r},j}.$$

Then

$$\sum_{i=1}^n C_{i,\bar{i},j} = \sum_{i \neq j} C_{i,\bar{i},j} = \sum_{i \neq j} \left(A_{i,\bar{i},j} - (-1)^{i+j} A_{i,\bar{j},i} - \frac{1}{n-1} \sum_{r=1}^n A_{r,\bar{r},j} \right) = (-1)^{j+1} \sum_{i=1}^n (-1)^i A_{i,\bar{j},i} = 0$$

by (iii), so that also (viii) holds.

Using (ii), we define $\{C_{i,\bar{j},\bar{k}}\}$. \square

Proof of Lemma 3.2.

- (i) is obvious from (3).
- (ii) Since Ω is real, $\Omega_{i,\bar{j}} = \overline{\Omega_{j,\bar{i}}}$, so that from (2) and (4) we get: $\forall i, j, k, A_{i,\bar{j},\bar{k}} = \overline{A_{j,\bar{i},k}}, C_{i,\bar{j},\bar{k}} = \overline{C_{j,\bar{i},k}$.
- (iii) Since Ω is closed, using (2) and (4) we get from (1): $\forall j, \sum_{i=1}^n (-1)^i A_{i,\bar{j},i} = \sum_{i=1}^n (-1)^i C_{i,\bar{j},i} = 0$.
- (iv) and (v) From (3) we get:

$$dz_k = dw_k + \sum_{r,s=1}^n b_{k,r,s} w_r dw_s + (\dots) = \sum_{s=1}^n (\delta_{k,s} + \beta_{k,s}) dw_s + (\dots),$$

where

$$\beta_{k,s} = \sum_{r=1}^n b_{k,r,s} w_r. \tag{5}$$

By a routine computation,

$$\widehat{dz}_i = \widehat{dw}_i + \left(\sum_{k=1, k \neq i}^n \beta_{k,k} \right) \widehat{dw}_i + \sum_{k=1, k \neq i}^n (-1)^{k+i-1} \beta_{k,i} \widehat{dw}_k + (\dots).$$

By (3),

$$\Omega_{i,\bar{j}}(z(w)) = \delta_{i,j} + \sum_{k=1}^n (A_{i,\bar{j},k} w_k + A_{i,\bar{j},\bar{k}} \overline{w}_k) + (\dots),$$

hence

$$\begin{aligned} \Omega &= \sigma_{n-1} \sum_{i,j=1}^n \left(\delta_{i,j} + \sum_{k=1}^n (A_{i,\bar{j},k} w_k + A_{i,\bar{j},\bar{k}} \overline{w}_k) + (\dots) \right) \\ &\quad \cdot \left(\widehat{dw}_i + \left(\sum_{k=1, k \neq i}^n \beta_{k,k} \right) \widehat{dw}_i + \sum_{k=1, k \neq i}^n (-1)^{k+i-1} \beta_{k,i} \widehat{dw}_k + (\dots) \right) \\ &\quad \wedge \left(\widehat{dw}_j + \left(\sum_{k=1, k \neq j}^n \beta_{k,k} \right) \widehat{dw}_j + \sum_{k=1, k \neq j}^n (-1)^{k+j-1} \beta_{k,j} \widehat{dw}_k + (\dots) \right). \end{aligned}$$

By (5) and some computation, we get:

$$\begin{aligned} (\sigma_{n-1})^{-1} \Omega &= \sum_{i=1}^n \left(1 + \sum_{k=1, k \neq i}^n \sum_{r=1}^n (b_{k,r,k} w_r + \overline{b_{k,r,k}} \overline{w}_r) + \sum_{k=1}^n (A_{i,\bar{i},k} w_k + A_{i,\bar{i},\bar{k}} \overline{w}_k) \right) \widehat{dw}_i \wedge \widehat{dw}_i \\ &\quad + \sum_{i,j=1, i \neq j}^n \left((-1)^{i+j-1} + \sum_{r=1}^n (b_{i,r,j} w_r + \overline{b_{j,r,i}} \overline{w}_r) + \sum_{k=1}^n (A_{i,\bar{j},k} w_k + A_{i,\bar{j},\bar{k}} \overline{w}_k) \right) \widehat{dw}_i \wedge \widehat{dw}_j \\ &\quad + (\dots). \end{aligned}$$

By (4) we get:

$$\begin{aligned} \Omega'_{i,\bar{i}}(w) &= 1 + \sum_{k=1}^n (C_{i,\bar{i},k} w_k + C_{i,\bar{i},\bar{k}} \overline{w}_k) + (\dots) \\ &= 1 + \sum_{k=1}^n \left(A_{i,\bar{i},k} + \sum_{r=1, r \neq i}^n b_{r,k,r} \right) w_k + \sum_{k=1}^n \left(A_{i,\bar{i},\bar{k}} + \sum_{r=1, r \neq i}^n \overline{b_{r,k,r}} \right) \overline{w}_k + (\dots), \end{aligned}$$

and, when $i \neq j$,

$$\begin{aligned} \Omega'_{i,\bar{j}}(w) &= \sum_{k=1}^n (C_{i,\bar{j},k} w_k + C_{i,\bar{j},\bar{k}} \overline{w}_k) + (\dots) \\ &= \sum_{k=1}^n (A_{i,\bar{j},k} + (-1)^{i+j-1} b_{i,k,j}) w_k + \sum_{k=1}^n (A_{i,\bar{j},\bar{k}} + (-1)^{i+j-1} \overline{b_{j,k,i}}) \overline{w}_k + (\dots). \end{aligned}$$

(vi) From (v) we get: $\forall i, k, C_{i,\bar{i},k} - A_{i,\bar{i},k} + b_{i,k,i} = \sum_{r=1}^n b_{r,k,r} := B_k$. Thus $\sum_{i=1}^n (C_{i,\bar{i},k} - A_{i,\bar{i},k}) + B_k = nB_k$, that is, $B_k = \frac{1}{n-1} \sum_{r=1}^n (C_{r,\bar{r},k} - A_{r,\bar{r},k})$.

(vii) and (viii) When i, k, j are distinct, $(-1)^{i+j} (A_{i,\bar{j},k} - C_{i,\bar{j},k}) = b_{i,k,j} = b_{i,j,k} = (-1)^{i+k} (A_{i,\bar{k},j} - C_{i,\bar{k},j})$, by (iv) and (i).

When $i \neq j$, from (vi) and (i), $(-1)^{i+j} (A_{i,\bar{j},i} - C_{i,\bar{j},i}) = b_{i,i,j} = b_{i,j,i} = A_{i,\bar{i},j} - C_{i,\bar{i},j} + \frac{1}{n-1} \sum_{r=1}^n (C_{r,\bar{r},j} - A_{r,\bar{r},j})$, so that $C_{i,\bar{i},j} = A_{i,\bar{i},j} - (-1)^{i+j} A_{i,\bar{j},i} + \frac{1}{n-1} (\sum_{r=1}^n C_{r,\bar{r},j} - \sum_{r=1}^n A_{r,\bar{r},j})$. \square

Remark. As regards the “uniqueness” of the coordinate system (w_1, \dots, w_n) considered in [Theorem 3.1](#), the question is: are there any coefficients left free among the $\{b_{k,r,s}\}$ in Formula (3)?

By [Lemma 3.2](#), we can turn the question to the coefficients $\{C_{i,\bar{j},k}\}$; in our setting, for distinct indices i, j, k , only $C_{i,\bar{j},k} \pm C_{i,\bar{k},j}$ is fixed, so that at least $\frac{n(n-1)(n-2)}{2}$ coefficients are free.

Proposition 3.3. *Let M be a complex n -dimensional manifold, and let ω be a real $(1, 1)$ -form on M . Then ω is associated with a balanced metric on M if and only if for every $p \in M$, there is a holomorphic coordinate system (w_1, \dots, w_n) centered at p such that*

$$\omega = \sigma_1 \sum_{i,j=1}^n \omega_{i,\bar{j}} dw_i \wedge d\bar{w}_j,$$

with

- (i) $\omega_{i,\bar{j}}(0) = \delta_{i,j}$,
- (ii) $\omega_{i,\bar{j}}(w)$ does not contain linear terms involving $w_i, w_j, \bar{w}_i, \bar{w}_j$,
- (iii) $d(\text{tr } \omega_{i,\bar{j}})(0) = 0$.

Proof. Put $\Omega := \omega^{n-1}$; then ω is associated with a balanced metric if and only if Ω is an $(n - 1)$ -Kähler form.

We shall use [Theorem 3.1](#) and its notation to prove [Proposition 3.3](#), and to prove also that the holomorphic coordinate system which is “good” for Ω is also “good” for ω .

First of all, notice that for every $p \in M$, $\Omega_{i,\bar{j}}(0) = \delta_{i,j} \iff \omega_{i,\bar{j}}(0) = \delta_{i,j}$.

Let us compare the coefficients of ω and Ω : $\omega = \sigma_1 \sum_{i,j=1}^n \omega_{i,\bar{j}} dz_i \wedge d\bar{z}_j$, with $\omega_{i,\bar{j}}(z) = \delta_{i,j} + \sum_{k=1}^n (a_{i,\bar{j},k} z_k + a_{i,\bar{j},\bar{k}} \bar{z}_k) + (\dots)$; and $\Omega = \sigma_{n-1} \sum_{i,j=1}^n \Omega_{i,\bar{j}} \widehat{dz}_i \wedge \widehat{d\bar{z}}_j$, with $\Omega_{i,\bar{j}}(z) = \delta_{i,j} + \sum_{k=1}^n (A_{i,\bar{j},k} z_k + A_{i,\bar{j},\bar{k}} \bar{z}_k) + (\dots)$.

A routine computation gives:

$$A_{i,\bar{i},k} = \sum_{r=1, r \neq i}^n a_{r,\bar{r},k}, \quad A_{i,\bar{j},k} = (-1)^{i+j-1} a_{j,\bar{i},k} \quad (j \neq i) \tag{6}$$

that is,

$$a_{i,\bar{i},k} = -A_{i,\bar{i},k} + \frac{1}{n-1} \sum_{r=1}^n A_{r,\bar{r},k}, \quad a_{j,\bar{i},k} = (-1)^{i+j} A_{i,\bar{j},k} \quad (j \neq i). \tag{7}$$

The condition $\partial\Omega|_p = 0$, i.e. $(\partial\omega \wedge \omega^{n-2})|_p = 0$, is given by: $\forall j, \sum_{i=1}^n (-1)^i A_{i,\bar{j},i} = 0$; using (6), it becomes: $\forall j, \sum_{i=1, i \neq j}^n a_{j,\bar{i},i} = \sum_{i=1, i \neq j}^n a_{i,\bar{i},j}$.

Let us choose the holomorphic coordinate system (w_1, \dots, w_n) centered at p as in [Theorem 3.1](#), and the corresponding constants $\{C_{i,\bar{j},k}, C_{i,\bar{j},\bar{k}}\}$ (in particular, it holds: $\forall j, \sum_{i=1}^n C_{i,\bar{i},j} = 0$).

With respect to coordinates (w_1, \dots, w_n) , we get $\omega = \sigma_1 \sum_{i,j=1}^n \omega'_{i,\bar{j}} dw_i \wedge d\bar{w}_j$, with

$$\omega'_{i,\bar{j}}(w) = \delta_{i,j} + \sum_{k=1}^n (c_{i,\bar{j},k} w_k + c_{i,\bar{j},\bar{k}} \bar{w}_k) + (\dots);$$

(6) and (7) become

$$c_{i,\bar{i},k} = -C_{i,\bar{i},k}, \quad c_{j,\bar{i},k} = (-1)^{i+j} C_{i,\bar{j},k} \quad (j \neq i). \tag{8}$$

This choice satisfies the thesis, because $\Omega_{i,\bar{j}}(0) = \delta_{i,j}$ implies (i), whereas (ii) is equivalent to $\forall i, j, c_{i,\bar{j},i} = c_{i,\bar{j},j} = 0$, and (iii) corresponds to $\forall j, \sum_{i=1}^n c_{i,\bar{i},j} = 0$.

But all this holds thanks to (8) and [Theorem 3.1](#). \square

Remark. Suppose we have chosen coordinates as in [Proposition 3.3](#). We can refine the Taylor expansion into

$$\omega'_{i,\bar{j}}(w) = \delta_{i,j} + \sum_{k=1}^n (c_{i,\bar{j},k} w_k + c_{i,\bar{j},\bar{k}} \bar{w}_k) + \sum_{k,r=1}^n (c_{i,\bar{j},k,\bar{r}} w_k \bar{w}_r + c_{i,\bar{j},k,r} w_k w_r + c_{i,\bar{j},\bar{k},\bar{r}} \bar{w}_k \bar{w}_r) + (\dots),$$

where (\dots) denotes now some terms of order at least three.

A computation of the Chern curvature tensor of the metric on the tangent bundle at p , carried out as in [3, Ch. VI, Theorem (4.8)], gives that

$$\Theta(TX)_p = \sum_{i,j,k,r=1}^n -c_{i,\bar{j},k,\bar{r}} dw_i \wedge d\bar{w}_j \otimes \partial_k^* \otimes \partial_r.$$

Since the Kähler condition is: $\partial_r \omega'_{i,\bar{j}} = \partial_i \omega'_{r,\bar{j}} \forall i, j, r$, in that case there is a simple choice of another coordinate system (u_1, \dots, u_n) such that any linear term vanishes, and moreover also holomorphic and anti-holomorphic second-order terms can be removed (as stated in [3], it is enough to choose $u_j = w_j + \frac{1}{3} \sum_{i,k,r=1}^n c_{i,\bar{j},k,\bar{r}} w_i w_k w_r$). In our case, since the closure condition is weaker, probably we can remove only a part of the holomorphic and anti-holomorphic second-order terms.

Corollary 3.4. ($n = 2$, see [4, pp. 107–108].) *Let M be a complex surface, and let ω be a real $(1, 1)$ -form on M . Then ω is associated with a Kähler metric if and only if for every $p \in M$, there is a holomorphic coordinate system (w_1, w_2) centered at p such that*

$$\omega = \sigma_1 \sum_{i,j=1}^2 \omega_{i,\bar{j}} dw_i \wedge d\bar{w}_j,$$

with

- (i) $\omega_{i,\bar{j}}(0) = \delta_{i,j}$,
- (ii) $\omega_{i,\bar{j}}(w)$ does not contain linear terms.

Proof. By Proposition 3.3, $\omega'_{i,\bar{j}}(w)$ does not contain linear terms when $i \neq j$; moreover,

$$c_{1,\bar{1},2} = -c_{2,\bar{2},2} = 0, \quad c_{2,\bar{2},1} = -c_{1,\bar{1},1} = 0,$$

hence we get the result. \square

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