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Partial differential equations

An example of non-decreasing solution for the KdV equation posed on a bounded interval

Un exemple de solution non décroissante de l'équation de KdV posée sur un intervalle borné

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A R T I C L E I N F O

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A B S T R A C T

An initial-boundary value problem for the KdV equation posed on a bounded interval is considered. The theory of Jacobi elliptic functions is used to obtain a new kind of stationary waves which are spatially periodic with a period equal to an interval length. The properties of those solutions are studied.

R É S U M É

On considère un problème avec donnée initiale et au bord pour l'équation de KdV posée sur un intervalle borné. La théorie des fonctions elliptiques de Jacobi est utilisée pour obtenir un nouveau type d'ondes stationnaires qui sont périodiques en espace avec une période égale à une longueur d'intervalle. Les propriétés de ces solutions sont étudiées.

1. Introduction

It is well known [1] that the KdV equation

$$v_t + vv_x + v_{xxx} = 0 \tag{1.1}$$

possesses spatially periodic cnoidal-wave solutions which are determined to be stable to a perturbation of the same period. They can be written explicitly as

$$v(x,t) = a + b\,\mathrm{cn}^2\big(d(x-ct);k\big) \tag{1.2}$$

in terms of the Jacobi elliptic function $\mathrm{cn}(x;k)$ where the elliptic modulus k and the parameters a, b, c and d are connected by a system of nonlinear transcendental equations (see [9]).

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Eq. (1.1) has been deduced to describe long waves of a small amplitude propagating in a dispersive media that occupies all the spatial domain ($x \in \mathbb{R}$). Numerical needs, however, require to cut-off the infinite domains of wave propagation [2]. The correct equation in this case (see, for instance, [2,12]) should be written as

$$v_t + v_x + v v_x + v_{xxx} = 0. \tag{1.3}$$

Once bounded intervals are considered as a spatial region of waves propagation, their lengths appear to be restricted by certain critical conditions. An important result in this context is the countable critical set (see *e.g.* [11]):

$$\mathcal{N} = \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2}; \quad k,l \in \mathbb{N}. \tag{1.4}$$

While studying the controllability and stabilization of solutions for (1.3), the set $\mathcal{N}$ provides qualitative difficulties when the length of a spatial interval coincides with some of its elements. In fact, the function

$$v(x) = 1 - \cos x$$

is a stationary (not decaying) solution for linearized (1.3) posed on $(0, 2\pi)$, and $2\pi \in \mathcal{N}$. However, if the transport term v_x is neglected, then (1.3) becomes (1.1), and the exponential decay rate of small solutions for (1.1) posed on any bounded interval is known to be held [7]. For (1.3) the same result has been shown if $L \notin \mathcal{N}$ (see [10]). The following questions arise:

- are there solutions of (1.3) which do not decay for $L \in \mathcal{N}$?
- if so, what is a "nonlinear analog" of $\mathcal{N}$?

Despite the valuable advances in [4–6], the question of whether solutions of undamped problems associated with nonlinear KdV decay as $t \to \infty$, for all finite interval lengths, is open (up to our knowledge).

In the present note, we construct explicitly the stationary solutions to homogeneous IBVP for nonlinear KdV (1.3) posed on a bounded interval $(0, L) \subset \mathbb{R}$ with some (critical) values of $L > 0$. These solutions clearly do not decay in time and can be viewed as nontrivial periodic solutions of (1.3) with spatial period L that are different from (1.2), as well as from the example of [8], where the authors have determined the existence of stationary solutions linked to Eq. (1.1) which is different from (1.3). Moreover, the authors do not provide how the solution depends on $L > 0$.

2. Main results

We start with the following result which guarantees the existence of explicit stationary solutions of the form:

$$v(x,t) = \phi(x) \tag{2.1}$$

related to the following initial boundary value problem:

$$v_t + v_x + v v_x + v_{xxx} = 0, \tag{2.2}$$
$$v(0,t) = v(L,t) = 0, \tag{2.3}$$
$$v_x(L,t) = 0, \tag{2.4}$$
$$v(x,0) = \phi(x). \tag{2.5}$$

Theorem 1. *For all $L \in (0, 2\pi)$, there exists a stationary solution $\phi \in C^\infty(\mathbb{R})$ satisfying*

(i) $\phi' + \frac{1}{2}(\phi^2)' + \phi''' = 0$, *in* $\mathbb{R}$,
(ii) $\phi(x+L) = \phi(x)$, $\forall x \in \mathbb{R}$,
(iii) $\phi(0) = \phi'(0) = 0$, $\phi''(0) \neq 0$.

Proof. In fact, let $L \in (0, 2\pi)$ be fixed. Substituting $v(x,t) = \phi(x)$ into (2.2)–(2.5) one has

$$\phi' + \frac{1}{2}(\phi^2)' + \phi''' = 0,$$
$$\phi(0) = \phi(L) = \phi'(L) = 0$$

which reads

$$\phi + \frac{1}{2}\phi^2 + \phi'' = A,$$
$$\phi(0) = \phi(L) = \phi'(L) = 0, \tag{2.6}$$

with an integration constant $A \in \mathbb{R}$. Two right-hand-side boundary conditions reduce (2.6) to be

$$\phi'^2 = \frac{1}{3}\left(-\phi^3 - 3\phi^2 + 6A\phi\right),$$
$$\phi(0) = 0. \tag{2.7}$$

Define the polynomial

$$F_A(y) = -y^3 - 3y^2 + 6Ay. \tag{2.8}$$

We are going to solve (2.7), provided that $F_A(y) \geqslant 0$ for y from a convenient interval to be determined. Moreover, since $\phi(0) = \phi(L) = \phi'(L) = 0$, it is natural to seek for solutions of (2.7) as for L-periodic solutions of (2.6).

Our aim now is to provide sufficient conditions on the value of $A \in \mathbb{R}$ in order to get periodic solutions. First, we assume F_A to be engaged with three distinct roots disposed as $\eta_1 < 0 < \eta_2$, that is

$$F_A(y) = (\eta_2 - y)(y - \eta_1)y.$$

Since

$$\eta_1 + \eta_2 = -3 \quad \text{and} \quad \eta_1\eta_2 = -6A,$$

one can assume $A > 0$. We discard the case $\eta_1 < \eta_2 < 0$ as not relevant for our purpose.

Solving $F_A = 0$, one get

$$\eta_2 = \frac{-3 + \sqrt{9 + 24A}}{2} \quad \text{and} \quad \eta_1 = \frac{-3 - \sqrt{9 + 24A}}{2}.$$

Aiming $F_A(y) \geqslant 0$, it holds

$$0 \leqslant \phi(x) \leqslant \eta_2$$

for all $x \in [0, L]$. Moreover, one has $\phi(L/2) = \eta_2$ as the maximum point of the solution.

Next, from (2.7) one has

$$\int_0^{\phi} \frac{\mathrm{d}y}{\sqrt{-y^3 - 3y^2 + 6Ay}} = \frac{1}{\sqrt{3}}(x + M), \quad \phi(0) = 0. \tag{2.9}$$

where $M \in \mathbb{R}$ is a constant of integration.

We solve Eq. (2.9) by using the theory of elliptic functions. Ref. [3] is strongly recommended to the reader for a more complete explanation of this subject.

Thus, we employ formula 236.00 from [3] to deduce the explicit solution ϕ of (2.6) as

$$\phi(x) = a\,\mathrm{sd}^2(bx; k), \tag{2.10}$$

where "sd" is the Jacobi elliptic function called "snoidal–dnoidal" ($\mathrm{sd} = \mathrm{sn}/\mathrm{dn}$) and $k \in (0, 1)$ is the modulus of the elliptic function. Here, parameters a, b and A are given in terms of the modulus k as

$$a = \frac{3k^2(1-k^2)(1-2k^2)}{1-4k^2+4k^4}, \qquad b = \frac{1}{2\sqrt{1-2k^2}}, \qquad A = \frac{3k^2(1-k^2)}{2(1-4k^2+4k^4)}.$$

Thus, the periodic solution ϕ becomes

$$\phi(x) = \left[\frac{3k^2(1-k^2)}{1-2k^2}\right]\mathrm{sd}^2\left(\frac{1}{2\sqrt{1-2k^2}}x; k\right). \tag{2.11}$$

Since the function sd^2 is $2K(k)$-periodic, a convenient expression for $L > 0$ with respect to k reads

$$L(k) = 4K(k)\sqrt{1-2k^2}. \tag{2.12}$$

Here

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

is the complete Jacobi elliptic integral of the first kind. Function $L(k)$ is strictly decreasing for $k \in (0, 1/\sqrt{2})$ with

$$\lim_{k\to 0^+} L(k) = 2\pi \quad \text{and} \quad \lim_{k\to(1/\sqrt{2})^-} L(k) = 0. \qquad \square$$

The graph of ϕ defined by (2.11) is visualized below for $L = 6$, see Fig. 1.

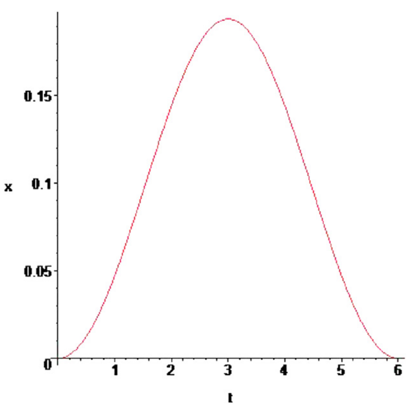

Fig. 1. The graph of ϕ from (2.10) for $L = 6$.

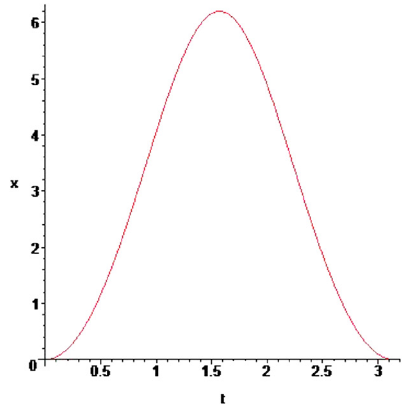

Fig. 2. The graph of ϕ from (2.10) for $L = \pi$.

Remark 2.1. The existence of periodic waves associated with (2.6) can also be determined by using a planar qualitative analysis of the general second-order differential equation

$$-u'' + g(u) = 0, \tag{2.13}$$

with a smooth function $g : \mathbb{R} \to \mathbb{R}$ which in our case reads

$$g(u) = -\frac{1}{2}u^2 - u + A.$$

Since $A > 0$, there are two distinct roots, say $u_1 = -1 - \sqrt{1+2A}$ and $u_2 = -1 + \sqrt{1+2A}$. Around u_2, one has the periodic orbits.

3. Comments

- Solution (2.11) defined for *all* $L \in (0, 2\pi)$ is an analog of $a(1 - \cos x)$, which solves linearized (1.3) completed by (2.3)–(2.5) with $L = 2\pi$ and an arbitrary $a \in \mathbb{R}$.
- In contrast to the linear case, the L^2-norm of ϕ cannot be "small" for small $L > 0$, as well as its amplitude.
- The periodicity of ϕ can be used to put forward explicit solutions related to the initial value problem (2.2)–(2.5) whose length L belongs to the critical set $\mathcal{N}$ in (1.4). In fact, in Theorem 1 the length of the interval must belong to the open set $(0, 2\pi)$. So, for problem posed on $[0, 2\pi]$, say, we can consider $L = \pi$ (see Fig. 2) in order to obtain π-periodic solutions. The required result is determined since every L-periodic function is also nL-periodic, for all $n \in \mathbb{N}$.
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