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A pseudo-differential calculus on the Heisenberg group

Un calcul pseudo-différentiel sur le groupe de Heisenberg

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ABSTRACT

In this note we present a symbolic pseudo-differential calculus on the Heisenberg group. We particularise to this group our general construction [4,2,3] of pseudo-differential calculi on graded groups. The relation between the Weyl quantisation and the representations of the Heisenberg group enables us to consider here scalar-valued symbols. We find that the conditions defining the symbol classes are similar but different to the ones in [1]. Applications are given to Schwartz hypoellipticity and to subelliptic estimates on the Heisenberg group.

RÉSUMÉ

Dans cette note, nous présentons un calcul pseudo-différentiel symbolique sur le groupe de Heisenberg. Nous particularisons à ce groupe notre construction générale [4,2,3] des calculs pseudo-différentiels sur les groupes de Lie gradués. Le lien entre la quantification de Weyl et les représentations du groupe de Heisenberg nous permet de considérer, dans ce cas, des symboles à valeurs scalaires. Nous trouvons que les conditions qui définissent les classes de symboles sont similaires, mais différentes de celles dans [1]. Comme applications, nous obtenons des résultats d'hypoellipticité de type Schwartz ainsi que des estimations subelliptiques sur le groupe de Heisenberg.

Version française abrégée

Dans [4], voir aussi [2,3], un calcul pseudo-différentiel est développé sur tout groupe gradué en utilisant les représentations de ces groupes. Ici, nous présentons nos résultats dans le cas particulier du groupe de Heisenberg $\mathbb{H}_n$.

Il est bien connu que les représentations de $\mathbb{H}_n$ sont intimement liées à la quantification de Weyl (voir [8] et la section 2 en anglais ci-dessous). Grâce à l'analogue de la quantification de Kohn–Nirenberg sur les groupes de Lie (voir par exemple [8, 6,4] et la section 4 en anglais ci-dessous), cela permet de développer un calcul pseudo-différentiel sur $\mathbb{H}_n$ avec symboles à valeurs scalaires qui dépendent de paramètres. Cependant, une difficulté persiste : celle de trouver les conditions que l'on doit imposer sur ces symboles pour que la classe d'opérateurs qui en résulte ait bien les propriétés attendues d'un calcul d'opérateurs.

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Bien que M. Taylor ait expliqué ces grandes lignes d'idées dans [8], son choix fut de restreindre son analyse dans [8] principalement au cas d'opérateurs invariants (c'est-à-dire de convolution) sur $\mathbb{H}_n$ avec des symboles définis par des développements asymptotiques. Jusqu'à ce jour, à la connaissance des auteurs de cette note, seule une étude porte sur des calculs non-invariants sur $\mathbb{H}_n$ avec symboles à valeurs scalaires : celle de H. Bahouri, C. Fermanian-Kammerer et I. Gallagher [1]. Leur travail traite uniquement du cas de $\mathbb{H}_n$. De plus, les conditions imposées sur leurs symboles peuvent sembler difficiles à appréhender aux yeux de certains lecteurs, car elles proviennent de parties techniques dans les démonstrations des propriétés du calcul (voir la version plus récente de [1] sur le serveur Hal).

Notre approche pour trouver les conditions à imposer sur les symboles est différente de celles de [8] et de [1] : nous particularisons au groupe de Heisenberg notre définition des calculs pseudo-différentiels, qui est valide sur une large classe de groupes de Lie nilpotents, plus précisément sur tout groupe gradué, voir [4,2,3]. Dans notre construction générale, les symboles sont à valeurs dans des classes d'opérateurs. Cependant, sur $\mathbb{H}_n$, grâce au lien entre la quantification de Weyl et les représentations de $\mathbb{H}_n$, cela est équivalent à utiliser des symboles à valeurs scalaires.

L'objet de cette note est de présenter ce que les conditions sur les symboles données dans le cas général [4,2,3] deviennent lorsqu'elles sont exprimées au niveau des symboles à valeurs scalaires dans le cas du groupe de Heisenberg $\mathbb{H}_n$. En particulier, nous trouvons des conditions similaires, mais différentes de celles de [1].

Dans la version française, nous donnons uniquement les conditions sur les symboles à valeurs scalaires. La version anglaise est plus complète : elle contient, en particulier, le lien entre les symboles à valeurs scalaires et opérationnelles ainsi que quelques applications de notre analyse. En effet, nous obtenons des résultats d'hypoellipticité de type Schwartz ainsi que des estimations subelliptiques sur le groupe de Heisenberg.

Nous réalisons le groupe de Heisenberg $\mathbb{H}_n$ comme la variété $\mathbb{R}^{2n+1}$ equipée de la loi de groupe

$$gg' = \left(x+x', y+y', t+t'+\frac{1}{2}\big(xy'-x'y\big)\right), \quad g=(x,y,t),\ g'=\big(x'y',t'\big)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}=\mathbb{R}^{2n+1}.$$

Nous adoptons ici la convention selon laquelle, si $x=(x_1,\dots,x_n)$ et $y=(y_1,\dots,y_n)$ sont deux vecteurs dans $\mathbb{R}^n$, alors $xy:=\sum_{j=1}^n x_jy_j$ est leur produit scalaire standard. La base canonique de l'algèbre de Lie $\mathfrak{h}_n$ de $\mathbb{H}_n$ est donnée par les champs de vecteurs invariants à gauche :

$$X_j=\partial_{x_j}-\frac{y_j}{2}\partial_t, \qquad Y_j=\partial_{y_j}+\frac{x_j}{2}\partial_t, \quad j=1,\dots,n, \text{ et } T=\partial_t.$$

Nous modifions la notation pour un indice $\alpha\in\mathbb{N}_0^{2n+1}$ pour écrire $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ avec :

$$\alpha_1=(\alpha_{1,1},\dots,\alpha_{1,n})\in\mathbb{N}_0^n, \qquad \alpha_2=(\alpha_{2,1},\dots,\alpha_{2,n})\in\mathbb{N}_0^n, \qquad \alpha_3\in\mathbb{N}_0.$$

Le degré d'homogénéité de α est $[\alpha]:=|\alpha_1|+|\alpha_2|+2\alpha_3$. Nous écrivons aussi $X^\alpha=X^{\alpha_1}Y^{\alpha_2}T^{\alpha_3}$, où $X^{\alpha_1}=X_1^{\alpha_{11}}\dots X_n^{\alpha_{1n}}$, et $Y^{\alpha_2}=Y_1^{\alpha_{21}}\dots Y_n^{\alpha_{2n}}$.

Nous utilisons la transformée de Fourier $\mathcal{F}_{\mathbb{R}^{2n+1}}=\mathcal{F}_{\mathbb{R}^N}$ sur $\mathbb{R}^N$ normalisée par :

$$\mathcal{F}_{\mathbb{R}^N}f(\xi)=(2\pi)^{-\frac{N}{2}}\int_{\mathbb{R}^N} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x \quad \big(\xi\in\mathbb{R}^N,\ f\in L^1\big(\mathbb{R}^N\big)\big),$$

et la quantification de Weyl Op^W, qui associe à un symbole raisonnable $a:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ l'opérateur défini par :

$$\mathrm{Op}^W(a)f(u)=a(D,X)f(u)=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(u-v)\xi}a\left(\xi,\frac{u+v}{2}\right)f(v)\,\mathrm{d}v\,\mathrm{d}\xi,$$

où $f\in\mathcal{S}(\mathbb{R}^n)$ et $u\in\mathbb{R}^n$.

Nous pouvons résumer la définition du calcul pseudo-différentiel sur le groupe de Heisenberg et certaines de ses propriétés sans faire référence au cas général des groupes gradués (voir la version en anglais pour cela) de la manière suivante.

Théorème 0.1. *Soit $1\geqslant\rho\geqslant\delta\geqslant0$ avec $(\rho,\delta)\neq(0,0)$. Pour tout $m\in\mathbb{R}$, considérons la classe d'opérateurs $\Psi^m_{\rho,\delta}(\mathbb{H}_n)$ définie dans [4] pour tout groupe gradué, particularisée ici au groupe de Heisenberg $\mathbb{H}_n$ muni de sa structure naturelle de groupe stratifié. De manière équivalente, $\Psi^m_{\rho,\delta}(\mathbb{H}_n)$ est définie par (10) dans la partie en anglais.*

(i) *Alors, $\Psi^m_{\rho,\delta}(\mathbb{H}_n)$ coïncide avec la classe d'opérateurs A sur $\mathbb{H}_n$ qui peuvent s'écrire sous la forme :*

$$A\varphi(g)=c_n'\int_{\lambda\in\mathbb{R}\backslash\{0\}}\mathrm{Tr}\big(\mathrm{Op}^W(a_{g,\lambda})\,\mathrm{Op}^W\big[\mathcal{F}_{\mathbb{R}^{2n+1}}\big(\varphi(g\,\cdot)\big)\big(\sqrt{|\lambda|}\,\cdot,\sqrt{\lambda}\,\cdot,\lambda\big)\big]\big)|\lambda|^n\,\mathrm{d}\lambda, \tag{1}$$

où $a=\{a(g,\lambda,\xi,u)=a_{g,\lambda}(\xi,u)\}$ est une fonction infiniment dérivable sur $\mathbb{H}_n\times\mathbb{R}\backslash\{0\}\times\mathbb{R}^n\times\mathbb{R}^n$ qui satisfait, pout tout $\alpha,\beta\in\mathbb{N}_0^n$, $\tilde\beta\in\mathbb{N}_0^{2n+1}$ et $\tilde\alpha\in\mathbb{N}_0$,

$$\big|\partial_\xi^\alpha\,\partial_u^\beta\,\tilde\partial^{\tilde\alpha}_{\lambda,\xi,u}\,X_g^{\tilde\beta}\,a_{g,\lambda}(\xi,u)\big|\leqslant C_{\alpha,\beta,\tilde\alpha,\tilde\beta}|\lambda|^{\rho\frac{|\alpha|+|\beta|}{2}}\big(1+|\lambda|\big(1+|\xi|^2+|u|^2\big)\big)^{\frac{m-2\rho\tilde\alpha+\delta[\tilde\beta]-\rho(|\alpha|+|\beta|)}{2}}. \tag{2}$$

Dans (1), g est un élément quelconque de $\mathbb{H}_n$ et φ est une fonction de la classe de Schwartz sur $\mathbb{H}_n \sim \mathbb{R}^{2n+1}$. Dans (2), $\tilde{\partial}_{\lambda,\xi,u}$ est l'opérateur $\tilde{\partial}_{\lambda,\xi,u} := \partial_\lambda - \frac{1}{2\lambda}\sum_{j=1}^n (u_j\partial_{u_j} + \xi_j\partial_{\xi_j})$, et $C_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}}$ est une constante indépendente de $(g,\lambda,\xi,u) \in \mathbb{H}_n \times \mathbb{R}\backslash\{0\} \times \mathbb{R}^n \times \mathbb{R}^n$.

(ii) *La classe d'opérateurs $\bigcup_{m\in\mathbb{R}} \Psi^m_{\rho,\delta}(\mathbb{H}_n)$ est donc un calcul pseudo-différentiel dans le sens où c'est une algèbre d'opérateurs stable pour l'adjonction $A \mapsto A^*$, qui contient le calcul différentiel invariant à gauche, et que chaque opérateur $A \in \Psi^m_{\rho,\delta}$ envoie continûment l'espace de Sobolev $L^2_s(\mathbb{H}_n)$ du groupe de Heisenberg dans $L^2_{s-m}(\mathbb{H}_n)$, la perte de dérivative étant donnée par l'ordre m de l'opérateur (pour tout $s \in \mathbb{R}$).*

Pour la définition des espaces de Sobolev $L^2_s(\mathbb{H}_n)$ sur le groupe de Heisenberg et plus généralement sur tout groupe stratifié, voir [5]. Pour la valeur de la constante c'_n dans (1), voir [4].

La partie (ii) ci-dessus résume certains résultats obtenus dans le cas général des groupes gradués dans [4].

1. Introduction

In [4], see also [2,3], a pseudo-differential calculus is developed in the setting of graded Lie groups using their representations. Here we present the results of this construction in the particular case of the Heisenberg group $\mathbb{H}_n$.

It is well known that the representations of $\mathbb{H}_n$ are intimately linked with the Weyl quantisation on $\mathbb{R}^n$ (see e.g., [8], and Section 2 below). Together with the analogue of the Kohn–Nirenberg quantisation on Lie groups (see e.g., [8,6,4], and Section 4 below), this link enables the development of pseudo-differential calculi on $\mathbb{H}_n$ with scalar-valued symbols that depend on parameters. However, the remaining difficulty lies in finding conditions to be imposed on those symbols so that the resulting class of operators has the expected properties of a calculus.

Although M. Taylor explained these general ideas in the setting of the Heisenberg groups in [8], he chose to restrict his analysis in [8] mainly to invariant (i.e. convolution) operators on $\mathbb{H}_n$ with symbols defined by some asymptotic expansions. To the authors' knowledge, the only study of non-invariant calculi with scalar-valued symbols on $\mathbb{H}_n$ was done, until now, by H. Bahouri, C. Fermanian-Kammerer and I. Gallagher in [1]. Their work is devoted to the case of $\mathbb{H}_n$ only. Moreover, the conditions imposed on the scalar-valued symbols might appear difficult to apprehend for some readers, as they come from technical parts of the proofs of the calculus properties (see the more recent version of [1] on the server Hal). Our conditions on symbol classes differ from those in [1] for small λ. At the end of this note, we list several applications of the analysis in our classes, to the hypoellipticity properties and subelliptic estimates for several operators on the Heisenberg group.

Our approach to find the conditions on the symbols is different from [8] and [1]: we particularise to the setting of $\mathbb{H}_n$ our definition of pseudo-differential calculi valid on a large class of nilpotent Lie groups, namely the graded groups, see [4, 2,3]. In our general construction, the symbols are operator-valued. Nonetheless on $\mathbb{H}_n$, using the link between the Weyl quantisation and the representations of $\mathbb{H}_n$, this is equivalent to using the scalar-valued symbols. The purpose of this note is to present what the general conditions on the symbols given in [4,2,3] become when expressed on the level of scalar-valued symbols of $\mathbb{H}_n$. In particular, we find conditions which are similar but different to the ones in [1]. As applications for our analysis, we give sufficient condition for Schwartz hypoellipticity and for subelliptic estimates on the Heisenberg group.

2. Schrödinger representations and Weyl quantisation

We start by fixing the notation required for presenting our results. We realise the Heisenberg group $\mathbb{H}_n$ as the manifold $\mathbb{R}^{2n+1}$ endowed with the law:

$$(x,y,t)(x',y',t') = \left(x+x', y+y', t+t' + \frac{1}{2}(xy'-x'y)\right),$$

where (x,y,t) and (x',y',t') are in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$. Here we adopt the convention that if x and y are two vectors in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, then xy denotes their standard scalar product:

$$xy = \sum_{j=1}^n x_j y_j \quad \text{if } x = (x_1,\dots,x_n),\ y = (y_1,\dots,y_n).$$

The canonical basis for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ is given by the left-invariant vector fields:

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \qquad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \quad j = 1,\dots,n, \text{ and } T = \partial_t.$$

The canonical commutation relations are:

$$[X_j, Y_j] = T, \quad j = 1,\dots,n,$$

and T is the centre of $\mathfrak{h}_n$. The Heisenberg Lie algebra is stratified via $\mathfrak{h}_n = V_1 \oplus V_2$ where V_1 is linearly spanned by the X_j's and Y_j's, while $V_2 = \mathbb{R}T$. Therefore, the group $\mathbb{H}_n$ is naturally equipped with the family of dilations D_r given by:

$$D_r(x,y,t) = r(x,y,t) = (rx, ry, r^2 t), \quad (x,y,t) \in \mathbb{H}_n,\ r > 0.$$

The 'canonical' positive Rockland operator in this setting is $\mathcal{R} = -\mathcal{L}$, where $\mathcal{L}$ is the sub-Laplacian:

$$\mathcal{L} := \sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right) = \sum_{j=1}^{n} \left(\left(\partial_{x_j} - \frac{y_j}{2}\partial_t\right)^2 + \left(\partial_{y_j} + \frac{x_j}{2}\partial_t\right)^2 \right).$$

The Schrödinger representations of the Heisenberg group $\mathbb{H}_n$ are the infinite dimensional unitary representations of $\mathbb{H}_n$ (we allow ourselves to identify unitary representations with their unitary equivalence classes). Parameterised by $\lambda \in \mathbb{R}\backslash\{0\}$, they act on $L^2(\mathbb{R}^n)$. We denote them by π_λ and realise them as:

$$\pi_\lambda(x, y, t)h(u) = \mathrm{e}^{\mathrm{i}\lambda(t+\frac{1}{2}xy)}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}yu}h\left(u + \sqrt{|\lambda|}x\right), \tag{3}$$

for $h \in L^2(\mathbb{R}^n)$, $u \in \mathbb{R}^n$, and $(x, y, t) \in \mathbb{H}_n$, where we use the convention:

$$\sqrt{\lambda} := \operatorname{sgn}(\lambda)\sqrt{|\lambda|} = \begin{cases} \sqrt{\lambda} & \text{if } \lambda > 0, \\ -\sqrt{|\lambda|} & \text{if } \lambda < 0. \end{cases} \tag{4}$$

The group Fourier transform of a function $\kappa \in L^1(\mathbb{H}_n)$ is by definition:

$$\widehat{\kappa}(\pi_\lambda) \equiv \pi_\lambda(\kappa) := \int_{\mathbb{H}_n} \kappa(x, y, t)\pi_\lambda(x, y, t)^* \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

As already noted in [8], it can be effectively computed by:

$$\pi_\lambda(\kappa)h(u) = \int_{\mathbb{R}^{2n+1}} \kappa(x, y, t)\mathrm{e}^{\mathrm{i}\lambda(-t+\frac{1}{2}xy)}\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}yu}h\left(u - \sqrt{|\lambda|}x\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,$$

for $h \in L^2(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$, that is,

$$\widehat{\kappa}(\pi_\lambda)(u) = \pi_\lambda(\kappa)h(u) = (2\pi)^{\frac{n}{2}}\mathrm{Op}^W\left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)\left(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\,\cdot, \lambda\right)\right]. \tag{5}$$

Here the Fourier transform $\mathcal{F}_{\mathbb{R}^{2n+1}} = \mathcal{F}_{\mathbb{R}^N}$ is defined via:

$$\mathcal{F}_{\mathbb{R}^N} f(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x \quad \left(\xi \in \mathbb{R}^N,\ f \in L^1\left(\mathbb{R}^N\right)\right),$$

and Op^W denotes the Weyl quantisation, which is given for a reasonable symbol a on $\mathbb{R}^n \times \mathbb{R}^n$, by:

$$\mathrm{Op}^W(a)f(u) = a(D, X)f(u) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(u-v)\xi} a\left(\xi, \frac{u+v}{2}\right) f(v)\,\mathrm{d}v\,\mathrm{d}\xi,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$. We keep the same notation π_λ for the corresponding infinitesimal representations. We readily compute that:

$$\begin{aligned} &\pi_\lambda(X_j) = \sqrt{|\lambda|}\partial_{u_j} = \mathrm{Op}^W\left(\mathrm{i}\sqrt{|\lambda|}\xi_j\right), \\ &\pi_\lambda(Y_j) = \mathrm{i}\sqrt{\lambda}u_j = \mathrm{Op}^W(\mathrm{i}\sqrt{\lambda}u_j), \\ &\pi_\lambda(T) = \mathrm{i}\lambda\,\mathrm{I} = \mathrm{Op}^W(\mathrm{i}\lambda), \end{aligned}$$

thus

$$\pi_\lambda(\mathcal{L}) = \sum_{j=1}^{n} \left(\pi_\lambda(X_j)^2 + \pi_\lambda(Y_j)^2\right) = |\lambda| \sum_{j=1}^{n} \left(\partial_{u_j}^2 - u_j^2\right) = -\mathrm{Op}^W\left(|\lambda| \sum_{j=1}^{n} \left(\xi_j^2 + u_j^2\right)\right).$$

With our choice of notation and definitions, the Plancherel measure is $c_n|\lambda|^n\,\mathrm{d}\lambda$ in the sense that the Plancherel formula:

$$\int_{\mathbb{H}_{n_o}} \left|\kappa(x, y, t)\right|^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = c_n \int_{\mathbb{R}\backslash\{0\}} \left\|\pi_\lambda(\kappa)\right\|_{\mathrm{HS}}^2 |\lambda|^n\,\mathrm{d}\lambda \tag{6}$$

holds for any $\kappa \in \mathcal{S}(\mathbb{H}_n)$. For the value of the constant c_n, see [4]. Here $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of an operator on $L^2(\mathbb{R}^n)$, that is, $\|B\|_{\mathrm{HS}}^2 = \mathrm{Tr}(B^*B)$. This allows one to extend unitarily the definition of the group Fourier transform to $L^2(\mathbb{H}_n)$. Formula (6) then holds for any $\kappa \in L^2(\mathbb{H}_n)$.

3. Difference operators

Difference operators were defined in [6,7] as acting on Fourier coefficients on compact Lie groups, and on graded nilpotent Lie groups in [4]. In the setting of the Heisenberg group, this yields the definition of the difference operators Δ_{x_j}, Δ_{y_j}, and Δ_t via:

$$\Delta_{x_j}\widehat{\kappa}(\pi_\lambda) := \pi_\lambda(x_j\kappa), \qquad \Delta_{y_j}\widehat{\kappa}(\pi_\lambda) := \pi_\lambda(y_j\kappa), \qquad \Delta_t\widehat{\kappa}(\pi_\lambda) := \pi_\lambda(t\kappa),$$

for suitable distributions κ defined on $\mathbb{H}_n$. We can compute that:

$$\Delta_{x_j}|_{\pi_\lambda} = \frac{1}{i\lambda}\,\mathrm{ad}\big(\pi_\lambda(Y_j)\big) = \frac{1}{\sqrt{|\lambda|}}\,\mathrm{ad}\,u_j,$$
$$\Delta_{y_j}|_{\pi_\lambda} = -\frac{1}{i\lambda}\,\mathrm{ad}\big(\pi_\lambda(X_j)\big) = -\frac{1}{i\sqrt{\lambda}}\,\mathrm{ad}\,\partial_{u_j},$$

and

$$\Delta_t|_{\pi_\lambda} = i\partial_\lambda + \frac{1}{2}\sum_{j=1}^{n}\Delta_{x_j}\Delta_{y_j}|_{\pi_\lambda} - \frac{i}{2\lambda}\sum_{j=1}^{n}\big(\pi_\lambda(Y_j)\Delta_{y_j}|_{\pi_\lambda} + \Delta_{x_j}|_{\pi_\lambda}\pi_\lambda(X_j)\big).$$

When $\pi_\lambda(\kappa) = \mathrm{Op}^W(a_\lambda)$ and $a_\lambda = \{a_\lambda(\xi,u)\}$, we have:

$$\left.\begin{aligned}
\Delta_{x_j}\pi_\lambda(\kappa) &= \mathrm{Op}^W\left(\frac{i}{\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda\right)\\
\Delta_{y_j}\pi_\lambda(\kappa) &= \mathrm{Op}^W\left(\frac{i}{\sqrt{\lambda}}\partial_{u_j}a_\lambda\right)\\
\Delta_t\pi_\lambda(\kappa) &= i\,\mathrm{Op}^W\big(\tilde{\partial}_{\lambda,\xi,u}a_\lambda(\xi,u)\big)
\end{aligned}\right\}, \tag{7}$$

where

$$\tilde{\partial}_{\lambda,\xi,u} := \partial_\lambda - \frac{1}{2\lambda}\sum_{j=1}^{n}(u_j\partial_{u_j} + \xi_j\partial_{\xi_j}). \tag{8}$$

For example, we have:

$$\Delta_{x_j}\pi_\lambda(Y_k) = \Delta_{x_j}\pi_\lambda(T) = \Delta_{y_j}\pi_\lambda(X_k) = \Delta_{y_j}\pi_\lambda(T) = 0,$$
$$\Delta_{x_j}\pi_\lambda(X_k) = \Delta_{y_j}\pi_\lambda(Y_k) = -\delta_{jk}\,\mathrm{I}, \qquad \Delta_t\pi_\lambda(T) = -\mathrm{I},$$
$$\Delta_{x_j}\pi_\lambda(\mathcal{L}) = -2\pi_\lambda(X_j), \qquad \Delta_{y_j}\pi_\lambda(\mathcal{L}) = -2\pi_\lambda(Y_j), \qquad \Delta_t\pi_\lambda(\mathcal{L}) = 0.$$

The following equalities shed some light on why, for example in [1], another normalisation of the Weyl symbol is preferred. Indeed, the expressions on the right-hand sides in (7), in particular for the operator $\tilde{\partial}_{\lambda,\xi,u}$ defined in (8), become very simple.

Lemma 3.1. *Let $a_\lambda = \{a_\lambda(\xi,u)\}$ be a family of Weyl symbols depending smoothly on $\lambda \neq 0$. If $\tilde{a}_\lambda$ is the renormalisation obtained via*

$$a_\lambda(\xi,u) := \tilde{a}_\lambda\big(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u\big),$$

then

$$\tilde{\partial}_{\lambda,\xi,u}a_\lambda(\xi,u) = \{\partial_\lambda\tilde{a}_\lambda\}\big(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u\big),$$
$$\frac{1}{i\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda = (\partial_{\xi_j}\tilde{a}_\lambda)\big(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u\big), \quad \textit{and} \quad \frac{1}{i\sqrt{\lambda}}\partial_{u_j}a_\lambda = (\partial_{u_j}\tilde{a}_\lambda)\big(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u\big).$$

Consequently,

$$\Delta_{x_j}\pi_\lambda(\kappa) = i\,\mathrm{Op}^W(\partial_{\xi_j}\tilde{a}_\lambda), \qquad \Delta_{y_j}\pi_\lambda(\kappa) = i\,\mathrm{Op}^W(\partial_{u_j}\tilde{a}_\lambda) \quad \textit{and} \quad \Delta_t\pi_\lambda(\kappa) = i\,\mathrm{Op}^W(\partial_\lambda\tilde{a}_\lambda).$$

4. Quantisation and symbol classes

In this note, for simplicity, we change slightly the notation with respect to the general case developed in [4]. Firstly we want to keep the letter x to denote part of the coordinates of the Heisenberg group and we choose to denote a general element of the Heisenberg group by, e.g.,

$$g = (x, y, t) \in \mathbb{H}_n.$$

Secondly we may define a symbol as parameterised by:

$$\sigma(g,\lambda) := \sigma(g,\pi_\lambda), \quad (g,\lambda) \in \mathbb{H}_n \times \mathbb{R}\backslash\{0\}.$$

Thirdly we modify the indices $\alpha \in \mathbb{N}_0^{2n+1}$ in order to write them as:

$$\alpha = (\alpha_1, \alpha_2, \alpha_3),$$

with

$$\alpha_1 = (\alpha_{1,1}, \dots, \alpha_{1,n}) \in \mathbb{N}_0^n, \qquad \alpha_2 = (\alpha_{2,1}, \dots, \alpha_{2,n}) \in \mathbb{N}_0^n, \qquad \alpha_3 \in \mathbb{N}_0.$$

The homogeneous degree of α is then:

$$[\alpha] = |\alpha_1| + |\alpha_2| + 2\alpha_3.$$

For each α we write:

$$g^\alpha = x^{\alpha_1} y^{\alpha_2} t^{\alpha_3}, \quad \text{where } x^{\alpha_1} = x_1^{\alpha_{11}} \dots x_n^{\alpha_{1n}},\ y^{\alpha_2} = y_1^{\alpha_{21}} \dots y_n^{\alpha_{2n}},$$

and we define the corresponding difference operator:

$$\Delta'^\alpha := \Delta_x^{\alpha_1} \Delta_y^{\alpha_2} \Delta_t^{\alpha_3}, \quad \text{where } \Delta_x^{\alpha_1} := \Delta_{x_1}^{\alpha_{11}} \dots \Delta_{x_n}^{\alpha_{1n}},\ \Delta_y^{\alpha_2} := \Delta_{y_1}^{\alpha_{21}} \dots \Delta_{y_n}^{\alpha_{2n}}.$$

We also write $X^\alpha = X^{\alpha_1} Y^{\alpha_2} T^{\alpha_3}$, where $X^{\alpha_1} = X_1^{\alpha_{11}} \dots X_n^{\alpha_{1n}}$, and $Y^{\alpha_2} = Y_1^{\alpha_{21}} \dots Y_n^{\alpha_{2n}}$.

Following [4], we define the symbol class $S^m_{\rho,\delta}(\mathbb{H}_n)$ as the set of symbols σ for which all the following quantities are finite:

$$\|\sigma\|_{S^m_{\rho,\delta},a,b,c} := \sup_{\lambda\in\mathbb{R}\backslash\{0\},\ g\in\mathbb{H}_n} \big\|\sigma(g,\lambda)\big\|_{S^m_{\rho,\delta},a,b,c}, \quad a,b,c \in \mathbb{N}_0,$$

where

$$\big\|\sigma(g,\lambda)\big\|_{S^m_{\rho,\delta},a,b,c} := \sup_{\substack{[\alpha]\leqslant a\\ [\beta]\leqslant b,\ |\gamma|\leqslant c}} \big\|\pi_\lambda(\mathrm{I}-\mathcal{L})^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{2}} X_g^\beta \Delta'^\alpha \sigma(g,\lambda)\pi_\lambda(\mathrm{I}-\mathcal{L})^{-\frac{\gamma}{2}}\big\|_{\mathrm{op}}.$$

A natural quantisation on any type-I Lie group is the analogue of the Kohn–Nirenberg quantisation on $\mathbb{R}^n$, see, e.g., [8] for general remarks, [6] for the consistent development in the case of compact Lie group, and [4] for the case of nilpotent Lie groups. In the particular case of the Heisenberg group, this quantisation associates with a symbol σ (for example in $S^m_{\rho,\delta}(\mathbb{H}_n)$) the operator $A = \mathrm{Op}(\sigma)$ acting on $\mathcal{S}(\mathbb{H}_n)$ given by:

$$A\varphi(g) = c_n \int_{\lambda\in\mathbb{R}\backslash\{0\}} \mathrm{Tr}\big(\pi_\lambda(g)\sigma(g,\lambda)\pi_\lambda(\varphi)\big)|\lambda|^n \,\mathrm{d}\lambda. \tag{9}$$

Here we have used our notation, especially for the Plancherel measure $c_n|\lambda|^n\,\mathrm{d}\lambda$, see (6). We denote by:

$$\Psi^m_{\rho,\delta}(\mathbb{H}_n) := \big\{\mathrm{Op}(\sigma),\ \sigma \in S^m_{\rho,\delta}(\mathbb{H}_n)\big\}, \tag{10}$$

the class of operators corresponding to the symbols in $S^m_{\rho,\delta}(\mathbb{H}_n)$ via this quantisation.

The main result of this note shows that the symbols $\sigma = \{\sigma(g,\lambda)\}$ in $S^m_{\rho,\delta}$ are all of the form $\sigma(g,\lambda) = \mathrm{Op}^W(a_{\lambda,g}(\xi,u))$ with $a_{\lambda,g}$ (called λ-*symbols*) satisfying properties of the Shubin type as follows.

Theorem 4.1. *Let ρ, δ be real numbers such that $1 \geqslant \rho \geqslant \delta \geqslant 0$ and $(\rho, \delta) \neq 0$.*

(i) *If $\sigma = \{\sigma(g,\lambda)\}$ is in $S^m_{\rho,\delta}(\mathbb{H}_n)$, then there exists a unique smooth function $a = \{a(g,\lambda,\xi,u) = a_{g,\lambda}(\xi,u)\}$ on $\mathbb{H}_n \times \mathbb{R}\backslash\{0\} \times \mathbb{R}^n \times \mathbb{R}^n$ such that:*

$$\sigma(g,\lambda) = \mathrm{Op}^W(a_{g,\lambda}). \tag{11}$$

It satisfies for any $\alpha, \beta \in \mathbb{N}_0^n$, $\tilde{\beta} \in \mathbb{N}_0^{2n+1}$ and $\tilde{\alpha} \in \mathbb{N}_0$,

$$\left|\partial_\xi^\alpha \partial_u^\beta \tilde{\partial}_{\lambda,\xi,u}^{\tilde{\alpha}} X_g^{\tilde{\beta}} a_{g,\lambda}(\xi,u)\right| \leqslant C_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}} |\lambda|^{\rho\frac{|\alpha|+|\beta|}{2}} \left(1+|\lambda|\left(1+|\xi|^2+|u|^2\right)\right)^{\frac{m-2\rho\tilde{\alpha}+\delta[\tilde{\beta}]-\rho(|\alpha|+|\beta|)}{2}}, \tag{12}$$

where the operator $\tilde{\partial}_{\lambda,\xi,u}$ was defined in (8).

(ii) *Conversely, if $a = \{a(g,\lambda,\xi,u) = a_{g,\lambda}(\xi,u)\}$ is a smooth function on $\mathbb{H}_n \times \mathbb{R}\backslash\{0\} \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying (12) for every $\alpha, \beta \in \mathbb{N}_0^n$, $\tilde{\alpha} \in \mathbb{N}_0$, then there exists a unique symbol $\sigma \in S^m_{\rho,\delta}(\mathbb{H}_n)$ such that (11) holds.*

(iii) *The resulting class of operators $\bigcup_{m\in\mathbb{R}} \Psi^m_{\rho,\delta}(\mathbb{H}_n)$ is an algebra of operators, the product being the usual composition. It is stable under taking the adjoint and contains the left-invariant differential calculus. Each operator $A \in \Psi^m_{\rho,\delta}(\mathbb{H}_n)$ maps continuously the Sobolev space $L^2_s(\mathbb{H}_n)$ of the Heisenberg group to $L^2_{s-m}(\mathbb{H}_n)$ with the loss of m derivatives (for any $s \in \mathbb{R}$).*

For the definition of the Sobolev spaces $L^2_s(\mathbb{H}_n)$ on $\mathbb{H}_n$ and more generally on any stratified group, see [5]. Part (iii) summarises the main results of the general construction made in [4] on any graded groups.

Let us notice that writing $\sigma(g,\lambda) = \mathrm{Op}^W(a_{g,\lambda})$ as in (11) and using (5) for:

$$\pi_\lambda(\varphi)\pi_\lambda(g) = \pi_\lambda\big(\varphi(g\,\cdot)\big)$$

yield the following alternative formula for the quantisation given in (9):

$$A\varphi(g) = c_n' \int_{\lambda\in\mathbb{R}\backslash\{0\}} \mathrm{Tr}\Big(\mathrm{Op}^W(a_{g,\lambda})\, \mathrm{Op}^W\big[\mathcal{F}_{\mathbb{R}^{2n+1}}\big(\varphi(g\,\cdot)\big)\big(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\,\cdot, \lambda\big)\big]\Big)|\lambda|^n \,\mathrm{d}\lambda. \tag{13}$$

For the value of the constant c_n', see [4]. The formula (13) now involves mainly 'Euclidean objects'.

5. Some applications

We finally note several applications of the above theorems to questions of hypoellipticity of (pseudo)-differential operators on the Heisenberg group. We say that a pseudo-differential operator A is *Schwartz hypoelliptic* whenever $f \in \mathcal{S}'(\mathbb{H}_n)$, $Af \in \mathcal{S}(\mathbb{H}_n)$ imply that $f \in \mathcal{S}(\mathbb{H}_n)$. Then, for example, as a simple consequence of our calculus, we obtain that the operator $\mathrm{I} - \mathcal{L}$ is Schwartz hypoelliptic. In fact, criteria can be given in terms of the λ-symbols.

Corollary 5.1. *Let $m \in \mathbb{R}$ and $1 \geqslant \rho \geqslant \delta \geqslant 0$, $\rho \neq 0$. Let $\sigma = \{\sigma(g,\lambda)\}$ be in $S^m_{\rho,\delta}(\mathbb{H}_n)$ with $\sigma(g,\lambda) = \mathrm{Op}^W(a_{g,\lambda})$ as in Theorem 4.1. Assume that there are $R \in \mathbb{R}$ and $C > 0$ such that, for any $(\xi,u) \in \mathbb{R}^{2n}$ and $\lambda \neq 0$ satisfying $|\lambda|(|\xi|^2+|u|^2) \geqslant R$, we have:*

$$\left|a_{g,\lambda}(\xi,u)\right| \geqslant C\left(1+|\lambda|\left(1+|\xi|^2+|u|^2\right)\right)^{\frac{m}{2}}. \tag{14}$$

Then the operator A in (9) (or, alternatively, in (13)) has a left parametrix, i.e. there exists $B \in \Psi^{-m}_{\rho,\delta}(\mathbb{H}_n)$ such that $BA - \mathrm{I} \in \Psi^{-\infty}$.

Corollary 5.1 has also a corresponding 'hypoellipticity version' that we omit here, but we give a few examples of both of them. First, let $m, m_o \in 2\mathbb{N}$ be two even integers such that $m \geqslant m_0$. Let A be a differential operator given by either $X^m + \mathrm{i}Y^{m_o} + T^{m_o/2}$ or $X^{m_o} + \mathrm{i}Y^m + T^{m_o/2}$ on $\mathbb{H}_1$. Then A is Schwartz hypoelliptic and satisfies the subelliptic estimates:

$$\forall s \in \mathbb{R},\ \exists C > 0,\ \forall f \in \mathcal{S}(\mathbb{H}_1), \quad \|f\|_{L^p_{s+m_o}(\mathbb{H}_1)} \leqslant C\|Af\|_{L^p_s(\mathbb{H}_1)}.$$

The above-mentioned conclusion that $\mathrm{I} - \mathcal{L}$ is Schwartz hypoelliptic can be also extended to variable coefficients using our calculus. For example, if f_1 and f_2 are complex-valued smooth functions on $\mathbb{H}_n$ such that:

$$\inf_{x\in\mathbb{H}_n,\, \lambda\geqslant\Lambda} \frac{|f_1(x)+f_2(x)\lambda|}{1+\lambda} > 0 \quad \text{for some } \Lambda \geqslant 0,$$

and such that functions $X^{\alpha_1} f_1$, $X^{\alpha_2} f_2$ are bounded for every $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$, then the differential operator $f_1(x) - f_2(x)\mathcal{L}$ is Schwartz-hypoelliptic and satisfies the following subelliptic estimates:

$$\forall s \in \mathbb{R},\ \exists C > 0,\ \forall \varphi \in \mathcal{S}(\mathbb{H}_n), \quad \|\varphi\|_{L^p_{s+2}(\mathbb{H}_n)} \leqslant C\|f_1\varphi - f_2\mathcal{L}\varphi\|_{L^p_s(\mathbb{H}_n)}.$$

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