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Partial differential equations

## Finite and infinite speed of propagation for porous medium equations with fractional pressure



*Vitesse de propagation finie et infinie pour des équations du milieu poreux avec une pression fractionnaire*

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## ABSTRACT

We study a porous medium equation with fractional potential pressure:

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla p), \quad p = (-\Delta)^{-s} u,$$

for  $m > 1$ ,  $0 < s < 1$  and  $u(x, t) \geq 0$ . To be specific, the problem is posed for  $x \in \mathbb{R}^N$ ,  $N \geq 1$ , and  $t > 0$ . The initial data  $u(x, 0)$  is assumed to be a bounded function with compact support or fast decay at infinity. We establish the existence of a class of weak solutions for which we determine whether, depending on the parameter  $m$ , the property of compact support is conserved in time or not, starting from the result of finite propagation known for  $m = 2$ . We find that when  $m \in [1, 2)$  the problem has infinite speed of propagation, while for  $m \in [2, \infty)$  it has finite speed of propagation. Comparison is made with other nonlinear diffusion models where the results are widely different.

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## R É S U M É

Nous étudions une équation du milieu poreux avec une pression potentielle fractionnaire:  $\partial_t u = \nabla \cdot (u^{m-1} \nabla p)$ ,  $p = (-\Delta)^{-s} u$ , pour  $m > 1$ ,  $0 < s < 1$  et  $u(x, t) \geq 0$ . Le problème se pose pour  $x \in \mathbb{R}^N$ ,  $N \geq 1$  et  $t > 0$ . La donnée initiale est supposée bornée avec support compact ou décroissance rapide à l'infini. Lorsque le paramètre  $m$  est variable, on obtient deux comportements différents comme suit : si  $m \in [1, 2)$ , le problème a une vitesse de propagation infinie, alors que pour  $m \in [2, \infty)$ , elle a une vitesse de propagation finie. On compare le résultat avec les comportements d'autres modèles de diffusion non linéaire, qui sont très différents.

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## Version française abrégée

Nous étudions un modèle de diffusion non linéaire avec pression non locale donné par :

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla p), \quad p = \mathcal{K}(u), \tag{1}$$

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pour  $m > 1$  et  $u(x, t) \geq 0$ . Le problème est posé pour  $x \in \mathbb{R}^N$ ,  $N \geq 1$ , et  $t > 0$ , et on se donne des conditions initiales :

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \quad (2)$$

où  $u_0 : \mathbb{R}^N \rightarrow [0, \infty)$  est bornée et à support compact ou à décroissance rapide à l'infini. La pression  $p$  est liée à  $u$  par un opérateur potentiel linéaire fractionnaire  $p = \mathcal{K}(u)$ , plus précisément  $\mathcal{K} = (-\Delta)^{-s}$  à  $0 < s < 1$ , qui s'exprime par le noyau  $K(x, y) = c|x - y|^{-(N-2s)}$  (c'est-à-dire un opérateur de Riesz).

Notre travail est motivé par deux travaux récents qui considèrent l'équation (1) pour  $m = 2$ . D'une part, Caffarelli et Vázquez étudient dans [4] le modèle de diffusion non linéaire de type milieu poreux avec des effets de diffusion non locaux :

$$\partial_t u = \nabla \cdot (u \nabla p), \quad p = (-\Delta)^{-s} u. \quad (CV)$$

L'étude est complétée dans les articles [5,6]. D'autre part, Head [12] a décrit la dynamique des dislocations dans les cristaux vues comme un continuum, et a proposé l'équation (1.1) avec  $s = 1/2$ ,  $m = 2$  si la dimension est  $N = 1$ . La densité de dislocations est de  $u = v_x$ , donc  $v$  résout le « problème intégré » :

$$v_t + |v_x|(-\partial_{xx})^{1-s} v = 0.$$

Le modèle a été récemment étudié sous cette forme par Biler, Karch et Monneau dans [3], où ils prouvent l'existence d'une unique solution de viscosité. Ils trouvent aussi une solution auto-similaire explicite, et ils décrivent le comportement asymptotique.

Notre note a pour but de mettre en clair la propriété de propagation finie pour les solutions du problème (1)–(2) en dépendance du paramètre  $m$ . On sait déjà que cette propriété est valable pour  $m = 2$ . On prouve ici qu'elle est encore valable pour  $m \in (2, \infty)$ , tandis que pour  $1 < m < 2$ , les solutions non négatives de cette équation sont strictement positives pour tout  $x \in \mathbb{R}$  et tout  $t > 0$ . Ce dernier résultat est prouvé rigoureusement en dimension  $N = 1$  et sous des conditions sur les données initiales. Notons finalement que les cas  $m = 1$  et  $m = 2$  étaient connus.

Tous ces points seront développés dans l'article [17].

## 1. Introduction

We study a nonlinear diffusion model with nonlocal pressure given by

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla p), \quad p = \mathcal{K}(u),$$

for  $m > 1$  and  $u(x, t) \geq 0$ . The problem is posed for  $x \in \mathbb{R}^N$ ,  $N \geq 1$ , and  $t > 0$ , and we give initial conditions

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N,$$

where  $u_0 : \mathbb{R}^N \rightarrow [0, \infty)$  is bounded with compact support or fast decay at infinity.

The pressure  $p$  is related to  $u$  through a linear fractional potential operator  $p = \mathcal{K}(u)$ . To be specific,  $\mathcal{K} = (-\Delta)^{-s}$  for  $0 < s < 1$  with kernel  $K(x, y) = c|x - y|^{-(N-2s)}$  (i.e. a Riesz operator). Moreover,  $\mathcal{K}$  is a self-adjoint operator with  $\mathcal{K} = \mathcal{H}^2$ .

Our work is motivated by two recent works that consider Eq. (1) for  $m = 2$ . On the one hand, Caffarelli and Vázquez have studied (1) for  $m = 2$  in [4] as a nonlinear diffusion equation of porous medium type with nonlocal diffusion effects:

$$\partial_t u = \nabla \cdot (u \nabla p), \quad p = (-\Delta)^{-s} u.$$

The study of this model has been pursued [5,6]. The properties of boundedness of the solutions,  $C^\alpha$  regularity, existence of self-similar solutions and asymptotic behavior are established. The latter paper points out that the application to problems in particle systems with long-range interactions [11] leads to more general equations of the form  $u_t = \nabla \cdot (f(u) \nabla \mathcal{K}g(u))$ , with convenient monotone functions  $f$  and  $g$ , thus motivating the interest in our present model.

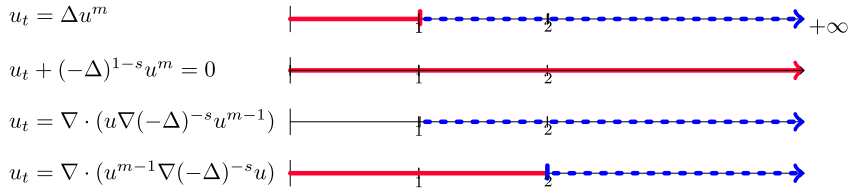
On the other hand, a similar model was proposed by Head [12] to describe the dynamics of dislocations in crystals seen as a continuum. When the space dimension is  $N = 1$ , the model becomes Eq. (1)  $s = 1/2$  and  $m = 2$ . The dislocation density is  $u = v_x$ , therefore  $v$  solves the “integrated problem”:

$$v_t + |v_x|(-\partial_{xx})^{1-s} v = 0.$$

The model in this form has been recently studied by Biler, Karch and Monneau in [3], where they prove existence of a unique viscosity solution.

## 2. Main results

We first propose a definition of the solution and establish the existence and main properties of the solutions.



**Fig. 1.** Ranges of exponent  $m \in (0, \infty)$ . Red line means infinite propagation, blue dotted line is finite propagation. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Fig. 1.** Intervalles de l'exposant  $m \in (0, \infty)$ . La ligne rouge signifie propagation infinie, la ligne pointillée bleue est la propagation finie. (Pour l'interprétation des couleurs dans cette figure on prie le lecteur de se référer à la version web de l'article.)

**Definition 1.** We say that  $u$  is a weak solution of (1)–(2) in  $Q_T = \mathbb{R}^N \times (0, T)$  with nonnegative initial data  $u_0 \in L^1(\mathbb{R}^N)$  if (i)  $u \in L^1(Q_T)$ , (ii)  $\mathcal{K}(u) \in L^1(0, T; W_{loc}^{1,1}(\mathbb{R}^N))$ , (iii)  $u^{m-1} \nabla \mathcal{K}(u) \in L^1(Q_T)$ , and (iv):

$$\int_0^T \int_{\mathbb{R}^N} u \phi_t \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} u^{m-1} \nabla \mathcal{K}(u) \nabla \phi \, dx \, dt + \int_{\mathbb{R}^N} u_0(x) \phi(x, 0) \, dx = 0 \tag{3}$$

holds for every test function  $\phi$  in  $Q_T$  such that  $\nabla \phi$  is continuous,  $\phi$  has compact support in  $\mathbb{R}^N$  for all  $t \in (0, T)$  and vanish near  $t = T$ .

Let state now the existence results as well as the basic properties for the solutions of problem (1).

**Theorem 1.** Let  $m \in (1, 2)$  (and  $s \in (0, 1/2)$  if  $N = 1$ ). Let  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Then there exists a weak solution  $u$  of Eq. (1) with initial data  $u_0$ . Moreover,  $u$  has the following properties:

1. (**Regularity**)  $u \in C([0, \infty) : L^1(\mathbb{R}^N))$ ,  $u \in L^\infty(\mathbb{R}^N \times (0, T))$ ,  $\nabla \mathcal{H}(u) \in L^2(Q)$ .
2. (**Conservation of mass**) For all  $t > 0$  we have  $\int_{\mathbb{R}^N} u(x, t) \, dx = \int_{\mathbb{R}^N} u_0(x) \, dx$ .
3. ( **$L^\infty$  estimate**) For all  $t > 0$ , we have  $\|u(\cdot, t)\|_\infty \leq \|u_0\|_\infty$ .
4. (**Energy estimate**) For all  $t > 0$ , the following estimate holds:

$$C \int_0^t \int_{\mathbb{R}^N} |\nabla \mathcal{H}(u)|^2 \, dx \, dt + \int_{\mathbb{R}^N} u(t)^{3-m} \, dx = \int_{\mathbb{R}^N} u_0^{3-m} \, dx, \quad C = (2 - m)(3 - m) > 0.$$

**Theorem 2.** Let  $m \in [2, \infty)$ . Let  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  be such that:

$$0 \leq u_0(x) \leq A e^{-a|x|} \quad \text{for some } A, a > 0. \tag{4}$$

Then there exists a weak solution  $u$  of Eq. (1) with initial data  $u_0$  such that  $u$  satisfies the properties 1, 2, 3 of Theorem 1. Moreover, the solution decays exponentially in  $|x|$  and enjoys suitable energy estimates, somewhat different from the ones of Theorem 1.

For  $m = 2$ , see [4]. The following is our most important contribution, which deals with the property of finite propagation of the solutions just constructed depending on the value of  $m$ .

**Theorem 3.** Let  $m \in (2, \infty)$ . The solution to problem (1)–(2) has the property of **finite speed of propagation** in the sense that, if  $u_0$  is compactly supported, then for any  $t > 0$ ,  $u(\cdot, t)$  is also compactly supported. On the other hand, if  $N = 1$ ,  $m \in (1, 2)$  and  $s \in (0, 1/2)$ , the solution  $u$  has **infinite speed of propagation** in the sense that even if the initial data is compactly supported, for any  $t > 0$  and any  $R > 0$ , the set  $\mathcal{M}_{R,t} = \{x : |x| \geq R, u(x, t) > 0\}$  has positive measure.

**Remarks.** (i) Finite propagation for the case  $m = 2$  has been proved in [4].

(ii) A main difficulty in the work to be done is the possible lack of uniqueness and comparison of the solutions, already noticed in [4]. This is reflected in the indirect statement of the last part of the theorem.

(iii) Since  $\nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u)$  tends to  $-\Delta^{1-s} u$  as  $m \rightarrow 1$ , Eq. (1) becomes  $u_t + (-\Delta)^{1-s} u = 0$ , known as the fractional heat equation, which has infinite speed of propagation. This propagation property is inherited by more general diffusion models as  $u_t + (-\Delta)^s u^m = 0$ , called the fractional porous medium equation, which has been studied in [9,10,20]. Therefore, a change in the behavior of the solutions for some  $m > 1$  was expected. We also motivate the result by numerical computations based on the scheme proposed by Teso and Vázquez in [18,19]. We give in the graph (see Fig. 1) a description of the result on finite propagation for different related models of nonlinear diffusion with or without fractional effects, see [21,10,3].

### 3. Existence of weak solutions

We make an approach to problem (1) based on regularization, elimination of the degeneracy and reduction of the spatial domain. Once we have solved the approximate problems, we derive estimates that allow us to pass to the limit in all the steps one by one, to finally obtain a weak solution of the original problem. In doing this, we follow the outline of [4]. Specifically, for small  $\epsilon, \delta, \mu \in (0, 1)$  and  $R > 0$ , we consider the following initial boundary value problem posed in  $Q_{T,R} = \{x \in B_R(0), t \in (0, T)\}$ :

$$\begin{cases} u_t = \delta \Delta u + \nabla((u + \mu)^{m-1} \nabla \mathcal{K}_\epsilon(u)) & \text{for } (x, t) \in Q_{T,R} \\ u(x, 0) = \widehat{u}_0(x) & \text{for } x \in B_R(0) \\ u(x, t) = 0 & \text{for } x \in \partial B_R(0), t \geq 0, \end{cases} \tag{5}$$

where  $\mathcal{K}_\epsilon$  is a smooth approximation of  $(-\Delta)^{-s}$ . Also,  $\mathcal{K}_\epsilon$  is a self-adjoint operator and we denote by  $\mathcal{K}_\epsilon = \mathcal{H}_\epsilon^2$ . The basic tool of the proof of existence is the following energy estimate for the approximated problem (5):

$$\int_{B_R} F_\mu(u(t)) \, dx + \delta \int_0^t \int_{B_R} \frac{|\nabla u|^2}{d_\mu(u)} \, dx \, dt + \int_0^t \int_{B_R} |\nabla \mathcal{H}_\epsilon(u)|^2 \, dx \, dt = \int_{B_R} F_\mu(\widehat{u}_0) \, dx \tag{6}$$

for all  $0 < t < T$ . The explicit formula for  $F_\mu$  is:

$$F_\mu(u) = \begin{cases} \frac{1}{(2-m)(3-m)} [(u + \mu)^{3-m} - \mu^{3-m}] - \frac{1}{2-m} \mu^{2-m} u, & \text{for } m \neq 2, 3 \\ -\log(1 + (u/\mu)) + u/\mu, & \text{for } m = 3 \\ (u + \mu) \log(1 + (u/\mu)) - u, & \text{for } m = 2. \end{cases}$$

For  $m = 2$  see [4].

#### 3.1. Uniform bounds in the case $m \in (1, 2)$

We obtain uniform bounds in all parameters  $\epsilon, R, \delta, \mu$  for the energy estimate (6), which allows us to pass to the limit and to obtain a solution of the original problem (1). More exactly,

$$\frac{1}{(m-2)(3-m)} \int_{\mathbb{R}^N} [(u_0 + \mu)^{3-m} - \mu^{3-m}] \leq \frac{(\|u_0\|_\infty + 1)^{2-m}}{m-2} \int_{\mathbb{R}^N} u_0 \, dx.$$

#### 3.2. Exponential tail in the case $m \in (2, \infty)$

This case is more delicate since the term  $\int_{B_R} F_\mu(u(t)) \, dx$  can not be easily uniformly controlled in  $\mu > 0$ .

In [4], when  $m = 2$ , the authors prove an exponential tail control of the approximate solution by using a comparison method with a suitable family of barrier functions, called true supersolutions. Their proof can be adapted to the case  $m \in (2, \infty)$  with a series of technical modifications caused by the power  $u^{m-1}$ .

We observe that for  $m \in (2, 3)$ :

$$\frac{1}{(2-m)(3-m)} \int_{\mathbb{R}^N} [(u(t) + \mu)^{3-m} - \mu^{3-m}] \, dx \leq \frac{1}{(2-m)(3-m)} \int_{\mathbb{R}^N} u(t)^{3-m} \, dx,$$

which is finite due to the tail control result.

### 4. Finite propagation for $m \in (2, \infty)$

Assume  $u$  is a bounded solution,  $0 \leq u \leq L$ , of Problem (1) as constructed before. Assume that  $u_0$  has compact support. Then we prove that  $u(\cdot, t)$  is compactly supported for all  $t > 0$ . More precisely, if  $u_0$  is below the “parabola-like” function,

$$U_0(x) = a(|x| - b)^2,$$

for some  $a, b > 0$ , with support in the ball  $B_b(0)$ , then for  $C$  is large enough, we prove that:

$$u(x, t) \leq a(Ct - (|x| - b))^2.$$

Since the problem does not satisfy the comparison principle, the proof of the above result is made by arguing at the first point in space and time where  $u(x, t)$  touches the parabola  $U$  from below. A contradiction is obtained as in the typical viscosity method if  $a$  and  $b$  are suitably chosen, even if there is generally no comparison principle.

### 5. Infinite propagation for $m \in (1, 2)$

The properties of the Model (1) in dimension  $N = 1$  can be obtained via a study of the properties of the integrated problem. We define  $v(x, t) = \int_{-\infty}^x u(y, t) dy$ . Then  $v$  is a solution to the problem:

$$\begin{cases} \partial_t v = -|v_x|^{m-1}(-\Delta)^\alpha v & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ v(x, 0) = v_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{7}$$

where the initial data is given by  $v_0(x) = \int_{-\infty}^x u_0(y) dy$ . The exponents  $\alpha$  and  $s$  are linked by:

$$\alpha = 1 - s. \tag{8}$$

The sketch of the proof of the infinite speed of propagation is as follows.

We define the notions of viscosity sub-solution, super-solution and solution in the sense of Crandall–Lions [8]. The definition will be adapted to our problem by considering the time dependency and also the nonlocal character of the fractional Laplacian operator.

The existence of a unique viscosity solution to Problem (7) follows as in [3]. The standard comparison principle for viscosity solutions also holds true (see [14,15]).

We give now our extended version of parabolic comparison principle, which represents an important instrument when using barrier methods. A similar parabolic comparison has been proved in [7] and has been used for instance in [7,16]. Let  $\Omega \subset \mathbb{R}$  possibly unbounded.

**Proposition 1.** *Let  $m > 1$ ,  $\alpha \in (0, 1)$ . Let  $v$  be a viscosity solution of problem (7). Let  $\Phi : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  such that  $\Phi \in C^2(\Omega \times (0, T))$ . Assume that:*

- $\Phi_t + |\Phi_x|^{m-1}(-\Delta)^\alpha \Phi < 0$  for  $x \in \Omega$ ,  $t \in [0, T]$ .
- $\Phi(x, 0) < v(x, 0)$  for all  $x \in \mathbb{R}^N$  (comparison at initial time).
- $\Phi(x, t) < v(x, t)$  for all  $x \in \mathbb{R}^N \setminus \Omega$  and  $t \in (0, T)$  (comparison on the parabolic boundary).

Then  $\Phi(x, t) \leq v(x, t)$  for all  $x \in \mathbb{R}$ ,  $t \in (0, T)$ .

We construct a lower barrier of the form:

$$\Phi_\epsilon(x, t) = (t + \tau)^{b\gamma} ( (|x| + \xi)^{-\gamma} + G(x) ) - \epsilon, \quad t \geq 0, x \in \mathbb{R}. \tag{9}$$

where  $\gamma = (2\alpha + m)/(2 - m)$  is the exponent of a kind of self-similar solution of this equation. The function  $G$  is such that, given any two constants  $C_1 > 0$  and  $C_2 > 0$ , we have that:

- (G1)  $G$  is compactly supported in the interval  $(-x_0, \infty)$ .
- (G2)  $G(x) \leq C_1$  for all  $x \in \mathbb{R}$ .
- (G3)  $(-\Delta)^s G(x) \leq -C_2|x|^{-(1+2s)}$  for all  $x < x_0$ .

We also prove the existence of this kind of function. To finish, we show that, and for any  $t_1 > 0$  and any  $x_1 \in \mathbb{R}$ , we have that  $\Phi_\epsilon$  satisfies the hypothesis of Proposition 1 and  $\Phi_\epsilon(x_1, t_1) > 0$  for a suitable choice of the parameters  $\epsilon, \xi$  and  $\tau$ . A similar lower barrier has been constructed in [16].

All these results will be developed in [17].

### 6. Comments and open problems

- *Explicit solutions.* Y. Huang reports [13] the explicit expression of the Barenblatt solution for the special value of  $m$ ,  $m_{ex} = (N + 6s - 2)/(N + 2s)$ . The profile is given by

$$F_M(y) = \lambda(R^2 + |y|^2)^{-(N+2s)/2}$$

where the two constants  $\lambda$  and  $R$  are determined by the total mass  $M$  of the solution and the parameter  $\beta$ . Note that for  $s = 1/2$ , we have  $m_{ex} = 1$ , and the solution corresponds to the linear case,  $u_t = (-\Delta)^{1/2}u$ ,  $F_{1/2}(r) = C(a^2 + r^2)^{-(N+1)/2}$ .

Different generalizations of model (CV) are worth studying:

- Changing-sign solutions for the problem  $\partial_t u = \nabla \cdot (|u| \nabla p)$ ,  $p = (-\Delta)^{-s}u$ .
- Starting from the problem (CV), an alternative is to consider the problem:

$$u_t = \nabla \cdot (|u| \nabla (-\Delta)^{-s}(|u|^{m-2}u)), \quad x \in \mathbb{R}^N, t > 0,$$

with  $m > 1$ . This problem has been studied by Biler, Imbert and Karch in [1,2]. They construct explicit compactly supported self-similar solutions that generalize the Barenblatt profiles of the PME. They do not prove the finite propagation of a general solution.

- We should consider combining the above models into  $\partial_t u = \nabla(|u|^{m-1} \nabla p)$ ,  $p = (-\Delta)^{-s} u$ .  
When  $s = 0$  and  $m = 2$  we obtain the signed porous medium equation  $\partial_t u = \Delta(|u|^{m-1} u)$ .

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