C. R. Acad. Sci. Paris, Ser. I 352 (2014) 31–34

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Probability theory

Variational principle for weighted porous media equation

Principe variationnel pour l'équation des milieux poreux

Alexandra Antoniouk [a], Marc Arnaudon [b]

[a] *Department of Nonlinear Analysis, Institute of Mathematics NAS Ukraine, Tereschchenkivska str. 3, Kyiv 01 601, Ukraine*
[b] *Institut de mathématiques de Bordeaux, CNRS UMR 5251, Université Bordeaux-1, 33405 Talence cedex, France*

A R T I C L E I N F O

Article history:
Received 26 August 2013
Accepted 22 November 2013
Available online 19 December 2013

Presented by the Editorial Board

A B S T R A C T

In this paper we state the variational principle for the weighted porous media equation. It extends V.I. Arnold's approach to the description of Euler flows as a geodesics on some manifold, i.e. as critical points of some energy functional.

R É S U M É

Dans cette article, on établit un principe variationnel pour l'équation des milieux poreux. On généralise ainsi la description de V.I. Arnold des flots d'Euler par des géodésiques vues comme des points critiques d'une fonctionnelle d'énergie.

1. Introduction

In the beginning of the 18th century, Leibniz, Maupertuis, and Euler claimed that all physical phenomenons might be obtained from the Least Action Principle, and since Lagrange and Hamilton it was well understood for the classical mechanics. However, it was only in 1966 that V.I. Arnold in [2] achieved it for the fluid dynamics. To do this, he remarked that the group of volume-preserving diffeomorphisms $\mathcal{D}_\mu(M)$ of a manifold M (μ being a given volume element on M) is the appropriate configuration space for the hydrodynamics of an incompressible fluid. In this framework, the solutions to the Euler equation become geodesic curves with respect to the right invariant metric on $\mathcal{D}_\mu$, which at $g \in \mathcal{D}_\mu$ is given by $(X, Y) = \int_M \langle X(x), Y(x)\rangle_x \, \mathrm{d}\mu(x)$, for $X, Y \in T_g\mathcal{D}_\mu$, $\langle\cdot,\cdot\rangle_x$ is a metric on T_xM, and μ is the volume element on M induced by the metric. The relation between geodesics on $\mathcal{D}_\mu$ and the Euler equation was further studied in [7] and shortly may be expressed in the following way. Let $t \mapsto g_t \in \mathcal{D}_\mu$ be a geodesic with respect to the right invariant metric $(\cdot,\cdot)$, $v_t = \frac{\mathrm{d}}{\mathrm{d}t} g_t$ be the corresponding velocity, and $u_t = v_t \circ g_t^{-1}$ be a time-dependent vector field on M. Then u_t is a solution to the Euler equation for a perfect fluid. In particular, the map $t \mapsto g_t$ defined on some time interval $[0, T]$ minimizes the energy functional:

$$S(g) = \frac{1}{2} \int_0^T \left(\int_M \left\| \frac{\mathrm{d}g_t}{\mathrm{d}t} \right\|^2 \mathrm{d}\mu(x) \right) \mathrm{d}t$$

and the Euler–Lagrange equations for this functional are precisely the Euler equation for perfect fluid.

E-mail addresses: antoniouk@imath.kiev.ua (A. Antoniouk), marc.arnaudon@math.u-bordeaux1.fr (M. Arnaudon).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.11.014

Developing this approach in [1,3], by means of stochastic methods, it was shown that an incompressible stochastic flow $g(u)$ with generator $\frac{1}{2}\Delta + u_t$ is critical for some energy functional if and only if u solves the Navier–Stokes equation for a viscous incompressible fluid. See also [4] and [8] for other stochastic characterizations of solutions to the Navier–Stokes equation. The purpose of this article is to show that the weighted porous media equation [6,5], which generalizes the standard porous media equation,

$$\frac{\partial u}{\partial t} = \left(-u \cdot \nabla + \frac{1}{2}\Delta\right)\left(\|u\|^{q-2}u\right) + \nabla P \tag{1}$$

may be also obtained in the framework of the Least Action Principle for a specially chosen energy functional. In the particular case of $q = 2$, this yields the Navier–Stokes equation.

2. Operator formulation of the variational principle

For simplicity, we work on the torus $\mathbb{T}$ of dimension N. From now on, when integrating in the torus, $\mathrm{d}x$ will stand for the normalized Lebesgue measure.

Definition 2.1. For some smooth divergence-free time-dependent vector field $(t, x) \mapsto v_t(x) \in T_x\mathbb{T}$, we define the flow of $\dot{v}_t$: $e_t(v) \in \mathcal{D}_\mu(\mathbb{T})$ as a solution to the ordinary differential equation:

$$\frac{\mathrm{d}e_t(v)}{\mathrm{d}t} = \dot{v}_t\big(e_t(v)\big), \qquad e_0(v) = \boldsymbol{I}_{\mathbb{T}}. \tag{2}$$

Let us remark that in some sense $e_t(v)$ is a perturbation of identity map in space $\mathcal{D}_\mu(\mathbb{T})$. The solvability of this equation easily follows from the compactness of $\mathbb{T}$ and the smoothness of v.

Consider a time-dependent divergence-free vector field u on $[0, T] \times \mathbb{T}$. So u takes its values in the tangent bundle of $\mathbb{T}$ which can at every point be identified with $\mathbb{R}^N$. "Divergence-free" means that $\sum_{j=1}^N \partial_j u^j \equiv 0$. Define the operator $L(u_t) : C^\infty(\mathbb{T}, \mathbb{R}^N) \to C^\infty(\mathbb{T}, \mathbb{R}^N)$ by $L(u_t)f = \frac{1}{2}\Delta f + u_t \cdot \nabla f$.

Definition 2.2. *The energy functional* is defined for $q > 1$ as

$$\mathcal{E}_q(u, v) = \frac{1}{q}\int_0^T \int_{\mathbb{T}} \big\|\big[\big(\partial_t + L(u_t)\big)e_t(v)\big]\big(e_t^{-1}(v)(x)\big)\big\|^q \,\mathrm{d}x\,\mathrm{d}t, \tag{3}$$

where $e_t^{-1}(v)$ is the inverse map of the diffeomorphism $e_t(v) : \mathbb{T} \to \mathbb{T}$.

Definition 2.3. We say that u is a critical point of $\mathcal{E}_q$ if for all divergence-free time-dependent vector field v such that $v_0 = 0$ and $v_T = 0$, $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\mathcal{E}_q(u, \varepsilon v) = 0$.

Theorem 1. *A divergence-free time-dependent vector field u is a critical point of $\mathcal{E}_q$, $q \geqslant 2$, if and only if there exists a function $P(x)$ such that* (1) *is satisfied.*

Proof. For $e_t(\varepsilon v)_*(u_t)(x) = T_{e_t^{-1}(\varepsilon v)(x)}e_t(\varepsilon v)(u_t(e_t^{-1}(\varepsilon v)(x)))$, we compute:

$$\big[\big(\partial_t + L(u_t)\big)e_t(\varepsilon v)\big]\big(e_t^{-1}(\varepsilon v)(x)\big) = \varepsilon\dot{v}\big(t, e_t^{-1}(\varepsilon v)(x)\big) + e_t(\varepsilon v)_*(u_t)(x) + \frac{1}{2}\big(\Delta e_t(\varepsilon v)\big)\big(e_t^{-1}(\varepsilon v)(x)\big),$$

where $T_y e_t(\varepsilon v)(\cdot)$ is the tangent map of $e_t(\varepsilon v)$ at point y. Therefore, we have:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\big[\big(\partial_t + L(u_t)\big)e_t(\varepsilon v)\big]\big(e_t^{-1}(\varepsilon v)(x)\big) = \dot{v}_t(x) + [u_t, v_t](x) + \frac{1}{2}\Delta v_t(x).$$

Since $u_t = (\partial_t + L(u_t))(\boldsymbol{I})$, for $\boldsymbol{I} = e_t(0) : \mathbb{T} \to \mathbb{T}$ the identity map, $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\mathcal{E}_q(u, \varepsilon v)$ equals:

$$\int_0^T \int_{\mathbb{T}} \big\|\big(\partial_t + L(u_t)\big)(\boldsymbol{I})\big\|^{q-2}\left\langle \dot{v}_t + [u_t, v_t] + \frac{1}{2}\Delta v_t, u_t\right\rangle \mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{T}} \|u_t\|^{q-2}\left\langle \dot{v}_t + [u_t, v_t] + \frac{1}{2}\Delta v_t, u_t\right\rangle \mathrm{d}x\,\mathrm{d}t.$$

On the other hand,

$$0 = \int_{\mathbb{T}} \|u_T\|^{q-2}\langle u_T, v_T\rangle\,\mathrm{d}x = \int_0^T \int_{\mathbb{T}} \big\{\|u_t\|^{q-2}\langle u_t, \dot{v}_t\rangle + \big\langle \|u_t\|^{q-4}(q-2)\langle \dot{u}_t, u_t\rangle u_t + \|u_t\|^{q-2}\dot{u}_t, v_t\big\rangle\big\}\,\mathrm{d}x\,\mathrm{d}t.$$

Therefore, writing $u = u_t$ and $v = v_t$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{E}_q(u, \varepsilon v) + \int_0^T \int_{\mathbb{T}} \left\{ \|u\|^{q-2} \left(\langle \dot{u}, v \rangle - \big\langle [u, v], u \big\rangle - \frac{\langle \Delta v, u \rangle}{2} \right) + (q-2)\|u\|^{q-4} \langle \dot{u}, u \rangle \langle u, v \rangle \right\} \mathrm{d}x\,\mathrm{d}t.$$

Due to equalities $\int_{\mathbb{T}} \|u\|^{q-2} \langle \nabla_v u, u \rangle \,\mathrm{d}x = \frac{1}{q} \int_{\mathbb{T}} \langle \nabla \|u\|^q, v \rangle \,\mathrm{d}x = -\frac{1}{q} \int_{\mathbb{T}} \|u\|^q \operatorname{div} v \,\mathrm{d}x = 0$ for $\operatorname{div} v = 0$, we have, using $[u, v] = \nabla_u v - \nabla_v u$,

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{E}_q\big(u, (\varepsilon v)\big) &= \int_0^T \int_{\mathbb{T}} \bigg\{ -\|u\|^{q-2} \langle \nabla_u v, u \rangle - \frac{1}{2} \big\langle v, \Delta\big(\|u\|^{q-2} u\big) \big\rangle \\
&\quad + (q-2)\|u\|^{q-4} \langle \dot{u}, u \rangle \langle u, v \rangle + \|u\|^{q-2} \langle \dot{u}, v \rangle \bigg\} \mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{T}} \left\langle \nabla_u\big(\|u\|^{q-2} u\big) - \frac{1}{2} \Delta\big(\|u\|^{q-2} u\big) + (q-2)\|u\|^{q-4} \langle \dot{u}, u \rangle u + \|u\|^{q-2} \dot{u}, v \right\rangle \mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{T}} \left\langle \left(\partial_t + u \cdot \nabla - \frac{1}{2}\Delta \right) \|u\|^{q-2} u, v \right\rangle \mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

(notice that in the second equality we used the fact that $\int_{\mathbb{T}} u(\langle v, \|u\|^{q-2} \rangle) \,\mathrm{d}x = \int_{\mathbb{T}} \operatorname{div} u \langle v, \|u\|^{q-2} \rangle \,\mathrm{d}x = 0$). This equality is true for all time-dependent divergence-free vector field v, so it gives the equivalence between u critical point of $\mathcal{E}_q$ and solution to equation (1). □

3. Stochastic variational principle for incompressible diffusion flows

We define a *diffusion flow* $g_t(x)$ on $\mathbb{T}$, $x \in \mathbb{T}$, $t \in [0, T]$, $T > 0$ as a stochastic process, which satisfies the Itô stochastic equation:

$$\mathrm{d}g_t(x) = \sigma\big(g_t(x)\big)\,\mathrm{d}W_t + u_t\big(g_t(x)\big)\,\mathrm{d}t, \quad g_0(x) = x \tag{4}$$

where u_t is a time-dependent vector field on $\mathbb{T}$, $\sigma \in \Gamma(Hom(\mathcal{H}, T\mathbb{T}))$ is a C^2-map satisfying, for all $x \in \mathbb{T}$, $(\sigma\sigma^*)(x) = \boldsymbol{I}_{T_x\mathbb{T}}$, W_t is a cylindric Brownian motion in Hilbert space $\mathcal{H}$.

Let us remark that a diffusion flow is a diffusion process $\{g_t(u)(x)\}_{t \geqslant 0}$ with generator $L(u_t) = \frac{1}{2}\Delta + u_t$. We define an *incompressible diffusion flow* $g_t(u)(x)(\omega)$ as a diffusion flow such that a.s. ω for all $t \geqslant 0$, the map $x \mapsto g_t(u)(x)(\omega)$ is a volume-preserving diffeomorphism of $\mathbb{T}$. Examples of incompressible diffusion flows can be found in [3]. Notice that a necessary condition is $\operatorname{div} u_t = 0$.

For the diffusion flow g_t (4), we define the *drift* as the time derivative of the finite variation part by $Dg_t(\omega) := u_t(g_t, \omega)$, and the *energy functional* by:

$$\mathcal{E}_q(g) := \frac{1}{q}\mathbb{E}\left[\int_0^T \int_{\mathbb{T}} \big\| Dg_t(x)(\omega) \big\|^q \,\mathrm{d}x\,\mathrm{d}t \right], \quad q > 1. \tag{5}$$

We make a perturbation by letting $g_t^v(u) = e_t(v) \circ g_t(u)$, where v is a smooth divergence-free time-dependent vector field and $e_t(v)$ is defined in (2).

Definition 3.1. We say that $g_t(u)$ is a critical point for the energy functional $\mathcal{E}_q$ if for all smooth time-dependent divergence-free vector field v on $T\mathbb{T}$ such that $v_0 = v_T = 0$, $\frac{d}{d\varepsilon}|_{\varepsilon=0} \mathcal{E}_q(g^{\varepsilon v}(u)) = 0$.

Theorem 2. *Let $q \geqslant 2$. An incompressible diffusion flow $g_t(u)$ with generator $L(u_t)$ is a critical point for the energy functional $\mathcal{E}_q$ if and only if there exists a function $P(x)$ such that u_t satisfies equation (1).*

Proof. The proof of this theorem is a consequence of Theorem 1 and the Itô formula. □

Acknowledgements

The research of A. Antoniouk was supported by grant No. 01-01-12 of the National Academy of Sciences of Ukraine (under the joint Ukrainian–Russian project of NAS of Ukraine and Russian Foundation of Basic Research).

This research was also supported by the French ANR's Grant No. ANR-09-BLAN-0364-01 ProbaGeo.

References

[1] M. Arnaudon, A.B. Cruzeiro, Lagrangian Navier–Stokes diffusions on manifolds: variational principle and stability, Bull. Sci. Math. 136 (8) (2012) 857–881.

[2] V.I. Arnold, Sur la géométrie difffentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fuides parfaits, Ann. Inst. Fourier 16 (1966) 316–361.

[3] F. Cipriano, A.B. Cruzeiro, Navier–Stokes equation and diffusions on the group of homeomorphisms of the torus, Commun. Math. Phys. 275 (1) (2007) 255–269.

[4] P. Constantin, G. Iyer, A stochastic Lagrangian representation of the three-dimensional incompressible Navier–Stokes equations, Commun. Pure Appl. Math. 61 (3) (2008) 330–345.

[5] J. Dolbeault, I. Gentil, A. Guillin, F.-Yu. Wang, L_q-Functional inequalities and weighted porous media equations, Potential Anal. 28 (2008) 35–59.

[6] J. Dolbeault, B. Nazaret, G. Savaré, On the Bakry–Emery criterion for linear diffusions and weighted porous media equations, Commun. Math. Sci. 6 (2) (2008) 477–494.

[7] D. Ebin, J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. Math. 92 (1) (1970) 102–163.

[8] G.L. Eyink, Stochastic least action principle for the incompressible Navier–Stokes equations, Physica D 239 (14) (2010) 1236–1240.