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Complex analysis

Subclasses of p-valent functions of bounded boundary rotation involving the generalized fractional differintegral operator

Sous-classes de fonctions p-valentes de rotation frontière bornée, relatives à l'opérateur différo-intégral fractionnaire généralisé

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A B S T R A C T

We introduce certain subclasses of p-valent functions of bounded boundary rotation involving the generalized fractional differintegral operator and investigate various inclusion relationships for these subclasses. Some interesting applications involving certain classes of integral operators are also considered.

R É S U M É

Nous introduisons certaines sous-classes de fonctions p-valentes de rotation frontière bornée, relatives à l'opérateur différo-intégral fractionnaire généralisé, et obtenons diverses relations d'inclusion de ces sous-classes. Quelques applications intéressantes impliquant certaines classes d'opérateurs intégraux sont également considérées.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad \left(p \in \mathbb{N} = \{1, 2, 3, \ldots\}\right), \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$ (see [5]).

Let $\mathcal{V}_{p,k}(\gamma)$ be the class of functions g analytic in $\mathbb{U}$ satisfying the properties $g(0) = p$ and

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$$\int_0^{2\pi}\left|\frac{\Re\{g(z)\}-\gamma}{p-\gamma}\right|\mathrm{d}\theta\leqslant k\pi\quad\left(z=r\mathrm{e}^{\mathrm{i}\theta},\ k\geqslant 2;\ 0\leqslant\gamma<p\right).\tag{2}$$

The class $\mathcal{V}_{p,k}(\gamma)$ was introduced by Aouf [1]. We note that:

(i) the class $\mathcal{V}_{1,k}(\gamma)=\mathcal{V}_k(\gamma)$ was introduced by Padmanabhan and Parvatham [10];
(ii) the class $\mathcal{V}_{1,k}(0)=\mathcal{V}_k$ was introduced by Pinchuk [12];
(iii) $\mathcal{V}_{p,2}(\gamma)=\mathcal{V}_p(\gamma)$ is the class of functions with positive real part greater than γ $(0\leqslant\gamma<p)$.

From (2), we have $g\in\mathcal{V}_{p,k}(\gamma)$ if and only if there exist $g_1,g_2\in\mathcal{V}_p(\gamma)$ such that:

$$g(z)=\left(\frac{k}{4}+\frac{1}{2}\right)g_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)g_2(z)\quad(z\in\mathbb{U}).\tag{3}$$

It is known [6] that the class $\mathcal{V}_k(\gamma)$ is a convex set.

Making use of $\mathcal{V}_{p,k}(\gamma)$, we introduce the subclasses $S_{p,k}(\gamma)$ and $C_{p,k}(\gamma)$, $0\leqslant\delta<p$, of $\mathcal{A}_p$ as follows:

$$S^*_{p,k}(\gamma)=\left\{f\in\mathcal{A}_p\colon\ \frac{zf'(z)}{f(z)}\in\mathcal{V}_{p,k}(\gamma);\ z\in\mathbb{U}\right\},\tag{4}$$

and

$$C_{p,k}(\gamma)=\left\{f\in\mathcal{A}_p\colon\ \frac{(zf'(z))'}{f'(z)}\in\mathcal{V}_{p,k}(\gamma);\ z\in\mathbb{U}\right\}.\tag{5}$$

We note that $S_{p,2}(\gamma)=S_p(\gamma)$ and $C_{p,2}(\gamma)=C_p(\gamma)$ $(0\leqslant\gamma<p)$, where $S_p(\gamma)$ and $C_p(\gamma)$ are, respectively, the classes of p-starlike functions of order γ and p-convex functions of order γ in $\mathbb{U}$ (see [8] and [11]).

Srivastava et al. [14] introduced the following generalized fractional integral and generalized fractional derivative operators as follows:

Definition 1.1. (See [14].) For real numbers $\lambda>0$, μ and η, the Saigo hypergeometric fractional integral operator: $I_{0,z}^{\lambda,\mu,\eta}:\mathcal{A}_p\to\mathcal{A}_p$ is defined by:

$$I_{0,z}^{\lambda,\mu,\eta}f(z)=\frac{z^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^z(z-t)^{\lambda-1}\,{}_2F_1\left(\lambda+\mu,-\eta;\lambda;1-\frac{t}{z}\right)f(t)\,\mathrm{d}t,\tag{6}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin, with the order $f(z)=O(|z|^{\varepsilon})$ $(z\to 0;\ \varepsilon>\max\{0,\mu-\lambda\}-1)$, and the multiplying of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 1.2. (See [14].) Under the hypotheses of Definition 1.1, the Saigo hypergeometric fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}:\mathcal{A}_p\to\mathcal{A}_p$ is defined by:

$$J_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\frac{1}{\Gamma(1-\lambda)}\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{\lambda-\mu}\int_0^z(z-t)^{-\lambda}\,{}_2F_1(\mu-\lambda,1-\eta;1-\lambda;1-\frac{t}{z})f(t)\,\mathrm{d}t\right\} & (0\leqslant\lambda<1),\\ \frac{\mathrm{d}^n}{\mathrm{d}z^n}J_{0,z}^{\lambda-\mu,\mu,\eta}f(z) & (n\leqslant\lambda<n+1;\ n\in\mathbb{N}),\end{cases}\tag{7}$$

where the multiplying of $(z-t)^{-\lambda}$ is removed as in Definition 1.1.

We note that $I_{0,z}^{\lambda,-\lambda,\eta}f(z)=D_z^{-\lambda}f(z)$ with $\lambda>0$ and $J_{0,z}^{\lambda,\lambda,\eta}f(z)=D_z^{\lambda}f(z)$ with $0\leqslant\lambda<1$, where $D_z^{-\lambda}$ denotes the fractional integral operator and D_z^{λ} denotes the fractional derivative operator studied by Owa [7].

Recently, Goyal and Prajapat [4] (see also [13]) introduced the generalized fractional differintegral operator $S_{0,z}^{\lambda,\mu,\eta}:\mathcal{A}_p\to\mathcal{A}_p$ $(p\in\mathbb{N};\ \eta\in\mathbb{R},\ \mu<p+1;\ z\in\mathbb{U})$ by:

$$S_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\frac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}z^{\mu}J_{0,z}^{\lambda,\mu,\eta} & (0\leqslant\lambda<\eta+p+1),\\ \frac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}z^{\mu}I_{0,z}^{-\lambda,\mu,\eta} & (-\infty<\lambda<0).\end{cases}\tag{8}$$

It is easily seen from a function f of the form (1), we have:

$$S_{0,z}^{\lambda,\mu,\eta} f(z) = z^p \, {}_3F_2(1, 1+p, 1+p+\eta-\mu; 1+p-\mu, 1+p+\eta-\lambda; z) * f(z)$$
$$= z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n} a_{n+p} z^{n+p} \quad (\mu < p+1;\ -\infty < \lambda < \eta+p+1), \tag{9}$$

where ${}_qF_s$ ($q \leqslant s+1$; $q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) is the well-known generalized hypergeometric function (see, for details, [9]) and $(v)_n$ is the Pochhammer symbol.

Upon setting

$$G_{p,\eta,\mu}^{\lambda}(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n} z^{n+p} \quad (\mu < p+1;\ -\infty < \lambda < \eta+p+1), \tag{10}$$

we define a new function $[G_{p,\mu,\eta}^{\lambda}(z)]^{-1}$ by means of the Hadamard product (or convolution):

$$G_{p,\eta,\mu}^{\lambda}(z) * \left[G_{p,\eta,\mu}^{\lambda}(z)\right]^{-1} = \frac{z^p}{(1-z)^{\delta+p}} \quad (\delta > -p;\ z \in \mathbb{U}). \tag{11}$$

Tang et al. [15] introduced the linear operator $H_{p,\eta,\mu}^{\lambda,\delta} : \mathcal{A}_p \to \mathcal{A}_p$ as follows:

$$H_{p,\eta,\mu}^{\lambda,\delta} f(z) = \left[G_{p,\eta,\mu}^{\lambda}(z)\right]^{-1} * f(z). \tag{12}$$

For $f \in \mathcal{A}_p$ given by (1), then from (12), we have:

$$H_{p,\eta,\mu}^{\lambda,\delta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\delta+p)_n(1+p-\mu)_n(1+p+\eta-\lambda)_n}{(1+p)_n(1+p+\eta-\mu)_n n!} a_{n+p} z^{n+p} \tag{13}$$

by using (13), we get:

$$z\big(H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)\big)' = (p+\eta-\lambda) H_{p,\eta,\mu}^{\lambda,\delta} f(z) - (\eta-\lambda) H_{p,\eta,\mu}^{\lambda+1,\delta} f(z) \tag{14}$$

and

$$z\big(H_{p,\eta,\mu}^{\lambda,\delta} f(z)\big)' = (\delta+p) H_{p,\eta,\mu}^{\lambda,\delta+1} f(z) - \delta H_{p,\eta,\mu}^{\lambda,\delta} f(z). \tag{15}$$

Next, using the operator $H_{p,\eta,\mu}^{\lambda,\delta}$, we introduce the subclasses $S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ and $C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ of $\mathcal{A}_p$ for $\mu, \eta \in \mathbb{R}$, $\mu < p+1$, $-\infty < \lambda < \eta+p+1$, $\delta > -p$, $p \in \mathbb{N}$, $k \geqslant 0$ and $0 \leqslant \gamma, \rho < p$:

$$S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) = \left\{f \in \mathcal{A}_p\colon H_{p,\eta,\mu}^{\lambda,\delta} f(z) \in S_{p,k}^{*}(\gamma)\ (z \in \mathbb{U})\right\}, \tag{16}$$

and

$$C_{p,k}^{\lambda,\delta}(p,\eta,\mu;\gamma) = \left\{f \in \mathcal{A}_p\colon H_{p,\eta,\mu}^{\lambda,\delta} f(z) \in C_{p,k}(\gamma)\ (z \in \mathbb{U})\right\}. \tag{17}$$

We also note that:

$$f \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \quad \Leftrightarrow \quad \frac{zf'}{p} \in C_{p,k}^{\lambda,\delta}(p,\eta,\mu;\gamma). \tag{18}$$

In this paper, we investigate several inclusion properties of the classes $S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ and $C_{p,k}^{\lambda,\delta}(p,\eta,\mu;\gamma)$ associated with the operator $H_{p,\eta,\mu}^{\lambda,\delta}$. Some applications involving integral operators are also considered.

2. Inclusion properties involving the operator $H_{p,\eta,\mu}^{\lambda,\delta}$

In order to prove the main results, we shall need the following lemmas.

Lemma 2.1. *(See [3].) Let h be convex univalent in $\mathbb{U}$ with $\Re\{\alpha h(z)+\beta\} > 0$ ($\alpha, \beta \in \mathbb{C}$). If p is analytic in $\mathbb{U}$ with $p(0) = h(0)$, then:*

$$p(z) + \frac{zp'(z)}{\alpha p(z)+\beta} \prec h(z) \quad \Longrightarrow \quad p(z) \prec h(z). \tag{19}$$

Theorem 2.2. *Let $k \geqslant 2$, $\delta \geqslant 0$ and $\eta \geqslant \lambda$. Then,*

$$S_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma) \subset S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \subset S_{p,k}^{\lambda+1,\delta}(\eta,\mu;\gamma).$$

Proof. First of all, we will show that $S_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma) \subset S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$. Let $f \in S_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma)$ and set:

$$g(z) = \frac{z(H_{p,\eta,\mu}^{\lambda,\delta} f(z))'}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)} \quad (z \in \mathbb{U}), \tag{20}$$

where the function g is analytic in $\mathbb{U}$ with $g(0) = p$. Using (15) and (20), we have:

$$\frac{z(H_{p,\eta,\mu}^{\lambda,\delta+1} f(z))'}{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)} = g(z) + \frac{zg'(z)}{g(z)+\delta} \in \mathcal{V}_{p,k}(\gamma) \quad (k \geqslant 2;\ 0 \leqslant \gamma < p;\ z \in \mathbb{U}). \tag{21}$$

Our aim is to show that $g(z) \in \mathcal{V}_{p,k}(\gamma)$. If $g(z) + \frac{zg'(z)}{g(z)+\delta} \in \mathcal{V}_{p,k}(\gamma)$, then there exist two functions $h_1, h_2 \in \mathcal{V}_p(\gamma)$ such that:

$$g(z) + \frac{zg'(z)}{g(z)+\delta} = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z).$$

Now let:

$$h_i(z) = g_i(z) + \frac{zg_i'(z)}{g_i(z)+\delta} \prec \frac{p+(p-2\gamma)z}{1-z} \quad (i = 1, 2;\ 0 \leqslant \gamma < p), \tag{22}$$

since $\delta \geqslant 0$, we see that:

$$\Re\left\{\frac{p+(p-2\gamma)z}{1-z} + \delta\right\} > 0 \quad (0 \leqslant \gamma < p;\ z \in \mathbb{U}).$$

Applying Lemma 2.1 to (22), it follows that:

$$g_i(z) \prec \frac{p+(p-2\gamma)z}{1-z} \quad (i = 1, 2;\ 0 \leqslant \gamma < p).$$

This means that $\Re(g_i(z)) > \gamma$, $i = 1, 2$. Now, if:

$$g(z) = \left(\frac{k}{4} + \frac{1}{2}\right)g_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)g_2(z),$$

then $g \in \mathcal{V}_{p,k}(\gamma)$, that is, $f \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$. To prove the second part, let $f \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ and put:

$$q(z) = \frac{z(H_{p,\eta,\mu}^{\lambda+1,\delta} f(z))'}{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)} \quad (z \in \mathbb{U}),$$

where q is analytic with $q(0) = p$. Then, by using the arguments similar to those detailed above with (14), it follows that $q \in \mathcal{V}_{p,k}(\gamma)$ in $\mathbb{U}$, which implies that $f \in S_{p,k}^{\lambda+1,\delta}(\eta,\mu;\gamma)$. This proves Theorem 2.2. □

Theorem 2.3. *Let $k \geqslant 2$, $\delta \geqslant 0$ and $\eta \geqslant \lambda$. Then,*

$$C_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma) \subset C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \subset C_{p,k}^{\lambda+1,\delta}(\eta,\mu;\gamma).$$

Proof. Applying (18) and Theorem 2.2, we observe that:

$$f \in C_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma) \iff \frac{zf'}{p} \in S_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma) \implies \frac{zf'}{p} \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \iff f \in C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma),$$

and

$$f \in C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \iff \frac{zf'}{p} \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \implies \frac{zf'}{p} \in S_{p,k}^{\lambda+1,\delta}(\eta,\mu;\gamma) \iff f \in C_{p,k}^{\lambda+1,\delta}(\eta,\mu;\gamma),$$

which evidently proves Theorem 2.3. □

3. Inclusion properties involving the integral operator $F_{p,c}$

In this section, we consider the generalized Libera integral operator $F_{p,c}(f)$ (see [2]) defined by:

$$F_{p,c}(f)(z) = \frac{c+p}{z^c} \int t^{c-1} f(z)\,\mathrm{d}t \quad (c > -p). \tag{23}$$

Theorem 3.1. *If $f \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$, then $F_{p,c}(f) \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ $(k \geqslant 2,\ c \geqslant 0)$.*

Proof. Let $f \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ and set:

$$g(z) = \frac{z(H_{p,\eta,\mu}^{\lambda,\delta} F_{p,c}(f)(z))'}{H_{p,\eta,\mu}^{\lambda,\delta} F_{p,c}(f)(z)} \quad (z \in \mathbb{U}), \tag{24}$$

where g is analytic in $\mathbb{U}$ with $g(0) = p$. From (23), we have:

$$z\big(H_{p,\eta,\mu}^{\lambda,\delta} F_{p,c}(f)(z)\big)' = (c+p) H_{p,\eta,\mu}^{\lambda,\delta} f(z) - c H_{p,\eta,\mu}^{\lambda,\delta} F_{p,c}(f)(z). \tag{25}$$

Then, by using (24) and (25), we obtain:

$$(c+p)\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{H_{p,\eta,\mu}^{\lambda,\delta} F_{p,c}(f)(z)} = g(z) + c. \tag{26}$$

Taking the logarithmic differentiation on both sides of (26) and multiplying by z, we have:

$$g(z) + \frac{zg'(z)}{g(z)+c} = \frac{z(H_{p,\eta,\mu}^{\lambda,\delta} f(z))'}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)} \in \mathcal{V}_{p,k}(\gamma) \quad (k \geqslant 2;\ 0 \leqslant \gamma < p;\ z \in \mathbb{U}). \tag{27}$$

Now the remaining part of Theorem 3.2 follows by employing the techniques of the first part that we used in proving Theorem 2.2 above. □

Next, we derive an inclusion property involving $F_{p,c}(f)$, which is given by the following theorem.

Theorem 3.2. *If $f \in C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$, then $F_{p,c}(f) \in C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ $(k \geqslant 2;\ c \geqslant 0)$.*

Proof. By applying Theorem 3.1, it follows that:

$$\begin{aligned} f \in C_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma) \quad &\Longleftrightarrow \quad \frac{zf'}{p} \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \quad \Longrightarrow \quad F_{p,c}\left(\frac{zf'}{p}\right) \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \\ &\Longleftrightarrow \quad \frac{z(F_{p,c}(f)(z))'}{p} \in S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \quad \Longleftrightarrow \quad F_{p,c}(f) \in C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma) \end{aligned}$$

which proves Theorem 3.2. □

Theorem 3.3. *The function f belongs to $S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ (or $C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$) if and only if χ defined by*

$$\chi(z) = \frac{\delta+p}{z^\delta} \int_0^z t^{\delta-1} f(t)\,\mathrm{d}t \quad (\delta > -p) \tag{28}$$

belongs to $S_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma)$ (or $C_{p,k}^{\lambda,\delta+1}(\eta,\mu;\gamma)$).

Proof. From (28), we have:

$$(\delta+p) f(z) = \delta\chi(z) + z\chi'(z). \tag{29}$$

Using (15) and (29), we can write:

$$(\delta+p) H_{p,\eta,\mu}^{\lambda,\delta} f(z) = \delta H_{p,\eta,\mu}^{\lambda,\delta}\chi(z) + z\big(H_{p,\eta,\mu}^{\lambda,\delta}\chi(z)\big)' = (\delta+p) H_{p,\eta,\mu}^{\lambda,\delta+1}\chi(z).$$

Therefore $H_{p,\eta,\mu}^{\lambda,\delta} f(z) = H_{p,\eta,\mu}^{\lambda,\delta+1}\chi(z)$ and this proves our result. □

Similarly we can prove the following theorem.

Theorem 3.4. *The function f belongs to $S_{p,k}^{\lambda+1,\delta}(\eta,\mu;\gamma)$ (or $C_{p,k}^{\lambda+1,\delta}(\eta,\mu;\gamma)$) if and only if ψ given by*:

$$\psi(z)=\frac{p+\eta-\lambda}{z^{\eta-\lambda}}\int_0^z t^{\eta-\lambda-1}f(t)\,\mathrm{d}t \quad (\eta>\lambda) \tag{30}$$

belongs to $S_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$ (or $C_{p,k}^{\lambda,\delta}(\eta,\mu;\gamma)$).

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