C. R. Acad. Sci. Paris, Ser. I 351 (2013) 645–648

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Automation (theoretical)

Growth bound of delay-differential algebraic equations

Taux de croissance des équations algébro-différentielles à retards

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ARTICLE INFO

Article history:
Received 7 November 2012
Accepted after revision 11 September 2013
Available online 10 October 2013

Presented by Olivier Pironneau

ABSTRACT

This paper deals with delay-differential algebraic equations, a large class of linear and finite-memory functional differential equations. We introduce several representations of delay operators that provide a simple definition for the concept of solutions of such systems. Then we study exponential solutions and prove that the rightmost zeros of a system characteristic function determine its growth bound.

RÉSUMÉ

Cet article traite des équations algébro-différentielles à retards, un large sous-ensemble des équations différentielles fonctionnelles linéaires à mémoire finie. Nous introduisons différentes représentations des opérateurs de retard, qui fournissent une définition simple du concept de solution de tels systèmes. Ensuite, nous étudions les solutions exponentielles et prouvons que les zéros les plus à droite de la fonction caractéristique d'un système déterminent son taux de croissance.

1. Introduction

Delay-differential algebraic equations (DDAE) are a class of functional differential equations (FDE) whose variables are connected through integrators and finite-memory delay operators. Such a system of equations, with variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and a finite memory length r, has the structure:

$$\left| \begin{aligned} \dot{x}(t) &= Ax_t + By_t \\ y(t) &= Cx_t + Dy_t \end{aligned} \right. \tag{1}$$

where z_t refers to the memory of the variable z at time t:

$$\operatorname{dom} z_t = [-r, 0] \quad \text{and} \quad \forall \theta \in [-r, 0],\ z_t(\theta) = z(t + \theta) \tag{2}$$

and the symbols A, B, C, D denote delay operators:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : C\big([-r, 0], \mathbb{C}^{n+m}\big) \to \mathbb{C}^{n+m}, \quad \text{linear and bounded.} \tag{3}$$

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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.08.001

This class of systems has been already considered in [1,2,4,10–12], but most of the literature has been focused on some of its subclasses: equations of retarded type, neutral type and difference equations. Nonetheless, the general model is important: it is required in the modelling of some physical phenomena such as lossless propagation (see [11] and the references therein) and in the control of dead-time systems when standard methods such as finite-spectrum assignment [9] are used.

A classic stability criterion determines the growth bound of a DDAE system from the location of the rightmost zeros of its characteristic function. The validity of this criterion has already been established with several methods under various restrictive assumptions: for systems of retarded type, for systems of neutral type, and difference equations whose difference operator combines only discrete and distributed delays [7] or satisfies a "jump" assumption [6], and for DDAE with discrete delays and stable difference operators [5].

We demonstrate in this paper that a single method, that combines the use of the Gearhart–Prüss theorem with bounds for the characteristic matrix inverse, established by complex analysis, can be used to prove this criterion in all these special cases. Actually, we require only the DDAE system to have a strictly causal difference operator, an assumption already used to ensure the well-posedness of the system. To the best knowledge of the author, this general result was not available.

Matrix-valued measures provide a concrete representation of delay operators: any linear bounded operator $L : C([-r,0],\mathbb{C}^j) \to \mathbb{C}^i$ corresponds to a unique countably additive function on the bounded Borel subsets of $\mathbb{R}$, supported on $[-r,0]$, with values in $\mathbb{C}^{i\times j}$. This alternate representation – that we still denote by L – is related to the initial operator by:

$$L\phi = \int \mathrm{d}L\,\phi := \sum_l \left[\int \phi_k\,\mathrm{d}L_{lk}\right] e_l \tag{4}$$

where $(e_1,\dots,e_i)$ denotes the canonical basis of $\mathbb{C}^i$. Let L^* be the measure obtained by symmetry around $t=0$ of L, such that for any bounded Borel set B, $L^*(B) = L(-B)$ and let $*$ be the convolution between time-dependent locally integrable functions – or more generally measures – of left-sided bounded support. The convolution of two scalar, vector or matrix-valued measures of compatible dimensions is defined as the combination of scalar convolution and linear algebra product; for example, for two matrix-valued measures A and B, $A*B$ is the matrix-valued measure such that $(A*B)_{ij} := \sum_k A_{ik} * B_{kj}$. We also implicitly extend functions defined on a subset of $\mathbb{R}$ by zero outside of their domain. With these conventions, for any continuous function z defined on $[-r,+\infty)$, we have $\forall t>0$, $Lz_t = (L^* * z)(t)$. As the right-hand side of this equation is still properly defined – as a locally integrable function of t – if z is merely locally integrable, we may rewrite Eq. (1) as a convolution equation.

Let e be the Heaviside function. We say that a pair of locally integrable functions (x,y), defined on $[-r,+\infty)$, with values in $\mathbb{C}^{n+m}$, is a (locally integrable) solution of (1) if there is an $f \in \mathbb{C}^n$ such that:

$$\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} e*A^* & e*B^* \\ C^* & D^* \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}(t) + \begin{bmatrix} f \\ 0 \end{bmatrix} \quad \text{for a.e. } t>0. \tag{5}$$

We assume in the sequel that the difference operator D is strictly causal, that is $D(\{0\}) = 0$. This condition ensures that this system of equations defines a well-posed initial value problem in the Hilbert product space $X = \mathbb{C}^n \times L^2([-r,0],\mathbb{C}^{n+m})$, see [1,2,12]: given any $(\phi,\chi,\psi) \in X$, there is a unique solution (x,y) such that $(x(0^+),x_0,y_0) = (\phi,\chi,\psi)$ and the mapping $(t \in \mathbb{R}_+ \mapsto \exp(\mathcal{A}t))$ given by $(x(t^+),x_t,y_t) = \exp(\mathcal{A}t)(\phi,\chi,\psi)$ for $t \geqslant 0$ is a strongly continuous semigroup on X.

2. Exponential solutions – characteristic matrix and resolvent operator

We denote by Δ the characteristic matrix of system (1), defined at any point $s \in \mathbb{C}$ by

$$\Delta(s) = \begin{bmatrix} sI_n & 0 \\ 0 & I_m \end{bmatrix} - \mathscr{L}\begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}(s) \tag{6}$$

where $\mathscr{L}$ is the Laplace transform.

The determinant of the characteristic matrix – the characteristic function – and its adjugate both have a quasi-polynomial structure:

$$\det\Delta(s) = \sum_{i=0}^{n} c_i(s)s^i, \qquad \operatorname{adj}\Delta(s) = \sum_{i=0}^{n} C_i(s)s^i \tag{7}$$

where the c_i (resp. C_i) are entire functions (resp. matrices of entire functions) bounded on any right-hand plane. Moreover, the leading coefficient of the characteristic function is given by $c_n(s) = \det\Delta_0(s)$, where Δ_0 is the characteristic matrix of the system $y(t) = Dy_t$. Lemma 2.1 establishes elementary properties of $\det\Delta_0$ and Lemma 2.2 describes how the zeros of $\det\Delta_0$ and $\det\Delta$ are connected.

For any real number σ, we denote by P_σ the open half-plane $\{s \in \mathbb{C} \mid \Re(s) > \sigma\}$ and for any positive η, we denote by Z_η the set of complex numbers whose distance to the zeros of $\det\Delta_0$ it at most η:

$$Z_\eta = \left\{ s \in \mathbb{C} \,\middle|\, \exists z \in \mathbb{C},\ \det\Delta_0(z) = 0 \wedge |s-z| \leqslant \eta \right\}. \tag{8}$$

Lemma 2.1 *(Zero clusters and lower bound). Let $\sigma \in \mathbb{R}$ and $\epsilon > 0$. There is an $\eta > 0$ such that any connected component Λ of the set Z_η is bounded and $\Lambda \subset P_{\sigma-\epsilon}$ if $\Lambda \cap P_\sigma \neq \emptyset$. Moreover, for any $\eta > 0$, there is a $\kappa > 0$ such that $|\det \Delta_0| \geqslant \kappa$ on $P_{\sigma-\epsilon} - Z_\eta$.*

Proof. Let Σ_m be the set of permutations of $\{1, \dots, m\}$ and

$$\det\nolimits_* M = \sum_{\sigma \in \Sigma_m} \operatorname{sgn}(\sigma) M_{1,\sigma(1)} * \cdots * M_{m,\sigma(m)}.$$

As $\Delta_0(s) = I_m - \mathscr{L} D^*(s)$, $\det \Delta_0 = \mathscr{L}\mu$ where $\mu = \det_*(\delta_0 I_m - D^*)$. The complex measure μ is a sum of convolution products of m complex measures supported on $[0, r]$, hence it is supported on $[0, mr]$. Consequently, $\det \Delta_0$ is an entire function that satisfies the inequality:

$$\left|\det \Delta_0(s)\right| \leqslant |\mu|\big([0, mr]\big) \max\big(1, \exp\big(-\Re(s)mr\big)\big). \tag{9}$$

Since $D(\{0\}) = 0$, we also have $\mu(\{0\}) = 1$, which yields:

$$\lim_{\Re s \to +\infty} \det \Delta_0(s) = 1. \tag{10}$$

The function $z \mapsto \det \Delta_0(iz)$ meets the assumptions of [8, th. VIII]. Thus, the number of distinct zeros $N(\rho)$ of $\det \Delta_0$ whose modulus is less than ρ is such that $\limsup_{\rho \to +\infty} N(\rho)/\rho \leqslant 2mr/\pi$. If there is an unbounded connected component of Z_η, there is a corresponding sequence $(z_n)_{n \in \mathbb{N}}$ of distinct zeros of $\det \Delta_0$ such that, for any $n \in \mathbb{N}$, $|z_{n+1} - z_n| \leqslant 2\eta$, which provides the density estimate $\limsup_{\rho \to +\infty} N(\rho)/\rho \geqslant 1/2\eta$. Consequently, if $\eta < \pi/4mr$, every connected component of Z_η is bounded.

The proof of the two remaining statements uses the same complex analysis argument. Consider a sequence s_n of numbers in $P_{\sigma-\epsilon}$ such that $\Re s_n \to x \in \mathbb{R}$ when $n \to +\infty$. From (9), it follows that the sequence of functions defined by $f_n(s) = \det \Delta_0(s + i\Im s_n)$ is locally uniformly bounded on $\mathbb{C}$. By Montel's theorem, a subsequence converges locally uniformly to an entire function f_∞, which by (10) is not identically zero. From Hurwitz's theorem, it follows that for any sufficiently small $\alpha > 0$, there is an arbitrarily large integer n such that $\det \Delta_0$ has m zeros in the open disk $B(s_n, \alpha)$, where m is the multiplicity of x if $f_\infty(x) = 0$, or 0 otherwise.

Consider a sequence Λ_n of bounded connected components of $Z_{1/2n}$, defined for n sufficiently large, such that $\Lambda_n \cap P_\sigma \neq \emptyset$ and $\Lambda_n - P_{\sigma-\varepsilon} \neq \emptyset$. For any such n and any $\alpha < \epsilon$, there is a $y_n \in \mathbb{R}$ such that the number of zeros of $\det \Delta_0$ in the open disk $B(\sigma + iy_n, \alpha)$ is greater than $n\alpha - 1$. This contradicts the result of the previous paragraph for the sequence $s_n = \sigma + iy_n$.

Finally, if $|\det \Delta_0|$ has no positive lower bound on $P_{\sigma-\epsilon} - Z_\eta$, we can find in this set a sequence s_n such that $\det \Delta_0(s_n) \to 0$ when $n \to +\infty$. By (10), the sequence s_n can be selected such that $\Re s_n$ has a limit $x \in \mathbb{R}$. As $f_\infty(x) = \lim_{n \to +\infty} f_n(s_n - i\Im s_n) = 0$, there is an integer n such that $\det \Delta_0$ has at least one zero in $B(s_n, \eta)$, a contradiction with the assumption that $s_n \notin Z_\eta$. $\square$

Lemma 2.2 *(Characteristic function zeros). Let $\sigma \in \mathbb{R}$. If the function $\det \Delta_0$ has an infinite number of zeros on P_σ, the function $\det \Delta$ has an infinite number of zeros on $P_{\sigma-\epsilon}$ for any $\epsilon > 0$.*

Proof. Suppose that $\det \Delta_0$ has an infinite number of zeros on P_σ. Let $\eta > 0$ be such that any connected component Λ of Z_η that contains such a zero is bounded and is included in $P_{\sigma-\epsilon}$ (Lemma 2.1). The zeros of $\det \Delta_0$ are isolated, hence every Λ contains a finite number of zeros, and the collection of sets Λ is therefore infinite. It is also locally finite: for any compact set $K \subset \mathbb{C}$ and any set Λ such that $K \cap \Lambda \neq \emptyset$, Λ contains a closed disk B_η of radius η such that $K \cap B_\eta \neq \emptyset$, hence $B_\eta \subset K' = K + \overline{B}(0, \eta)$ and only a finite number of such disjoint disks B_η may be contained in K'. Thus, for any $\rho > 0$, there is a set Λ that does not intersect $\overline{B}(0, \rho)$.

The lower bound from Lemma 2.1 and the quasi-polynomial structure of $\det \Delta_0$ provide the existence of a $\rho_0 > 0$ such that, if $s \in P_{\sigma-\epsilon}$ is not in $Z_{\eta/2}$ and satisfies $|s| > \rho_0$, then $|\det \Delta(s) - s^n \det \Delta_0(s)| < |s^n \det \Delta_0(s)|$. Let Λ_0 be one of the sets Λ that does not intersect $\overline{B}(0, \rho_0)$. As Λ_0 is included in $P_{\sigma-\epsilon}$, the application of Rouché's theorem to its boundary yields the existence of at least one zero of $\det \Delta$ in $P_{\sigma-\epsilon}$. We may more generally define $\rho_{n+1} = \sup\{|s|, s \in \Lambda_n\}$ and apply the same argument to a set Λ_{n+1} that does not intersect $\overline{B}(0, \rho_{n+1})$ to prove the existence of an infinite sequence of zeros of $\det \Delta$ in $P_{\sigma-\epsilon}$. $\square$

The infinitesimal generator $\mathcal{A}$ of the DDAE semigroup is defined by $\mathcal{A}(\phi, \chi, \psi) = (A\chi + B\psi, \dot{\chi}, \dot{\psi})$ on the domain $\{(\phi, \chi, \psi) \in \mathbb{C}^n \times W^{1,2}([-r, 0], \mathbb{C}^{n+m}) \mid \chi(0) = \phi,\ \psi(0) = C\chi + D\psi\}$, see [12]. The resolvent operator $(sI - \mathcal{A})^{-1}$ exists iff $\Delta(s)^{-1}$ exists: the resolvent set of $\mathcal{A}$ is:

$$\rho(\mathcal{A}) = \big\{s \in \mathbb{C}, \ker \Delta(s) = \{0\}\big\}. \tag{11}$$

Moreover, for any real number σ, there are constants κ_σ and λ_σ such that:

$$\big\|(sI - \mathcal{A})^{-1}\big\| \leqslant \kappa_\sigma \big\|\Delta(s)^{-1}\big\| + \lambda_\sigma \quad \text{if } \Re s \geqslant \sigma \text{ and } s \in \rho(\mathcal{A}). \tag{12}$$

Let the growth bound ω_0 of the DDAE system be the infimum of the real numbers ω for which there exists $\alpha > 0$ such that, for any initial condition $(x(0^+), x_0, y_0) \in X$, the solution of the system satisfies:

$$\forall t \geqslant 0, \quad \left\| \left(x\left(t^+\right), x_t, y_t\right)\right\|_X \leqslant \alpha \exp(\omega t) \left\| \left(x\left(0^+\right), x_0, y_0\right)\right\|_X. \tag{13}$$

Theorem 2.3 *(Growth bound). The growth bound of a DDAE system such that* $D(\{0\}) = 0$ *is determined by the rightmost zeros of the characteristic function and given by*:

$$\omega_0 = \sup\left\{\Re(s),\ s \in \mathbb{C} \;\middle|\; \det \Delta(s) = 0\right\}. \tag{14}$$

As a corollary, such a DDAE system is uniformly exponentially stable iff the set of zeros of its characteristic function is on the left of – and at a positive distance from – the imaginary axis.

Proof. We use the Gearhart–Prüss theorem [3] to establish the result: we prove that for any $\sigma > s(\mathcal{A})$, where $s(\mathcal{A})$ is the spectral bound of $\mathcal{A}$, $\|(sI - \mathcal{A})^{-1}\|$ is bounded on $\overline{P}_\sigma$; this is achieved by combining inequality (12) with the derivation of a bound for $\|\Delta(s)^{-1}\|$ on $\overline{P}_\sigma$.

The quasi-polynomial structure of the adjugate matrix yields on $\overline{P}_\sigma$ the estimate $\|\operatorname{adj} \Delta(s)\| \leqslant \kappa(1 + |s|^n)$. From (11) we deduce that $\det \Delta$ has no zero on $P_{\sigma-\epsilon}$ for any $\epsilon > 0$ such that $s(\mathcal{A}) < \sigma - \epsilon$ and hence, by Lemma 2.2, $\det \Delta_0$ has at most a finite number of zeros on $P_{\sigma-\epsilon/2}$ and therefore on $\overline{P}_\sigma$. On $\overline{P}_\sigma$ and away from these zeros, $|\det \Delta_0|$ has a positive lower bound κ' by Lemma 2.1. It follows from the quasi-polynomial structure of $\det \Delta$ that $|\det \Delta(s)| \geqslant \kappa''(1 + |s|^n)$ for a $\kappa'' > 0$, on $\overline{P}_\sigma$ except on a compact set K and by continuity, this estimate still holds on all of $\overline{P}_\sigma$ with a possibly smaller κ''. Finally, for any $s \in \overline{P}_\sigma$, $\|\Delta(s)^{-1}\| = \|\operatorname{adj} \Delta(s)\|/|\det \Delta(s)| \leqslant \kappa/\kappa''$. □

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