C. R. Acad. Sci. Paris, Ser. I 351 (2013) 635–638

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential Geometry

Regularity of the Kähler–Ricci flow

Régularité du flot de Kähler–Ricci

Gang Tian [a,b], Zhenlei Zhang [c]

[a] *BICMR, Peking University, Yiheyuan Road 5, Beijing 100871, China*
[b] *Department of Mathematics, Princeton University, NJ 02139, USA*
[c] *Capital Normal University, Xisanhuan North Road 105, Beijing 100048, China*

ARTICLE INFO

Article history:
Received 9 April 2013
Accepted 3 July 2013
Available online 5 September 2013

Presented by Jean-Pierre Demailly

ABSTRACT

In this short note, we announce a regularity theorem for the Kähler–Ricci flow on a compact Fano manifold (Kähler manifold with positive first Chern class) and its application to the limiting behavior of the Kähler–Ricci flow on Fano 3-manifolds. Moreover, we also present a partial C^0 estimate of the Kähler–Ricci flow under the regularity assumption, which extends previous works on Kähler–Einstein metrics and shrinking Kähler–Ricci solitons. The detailed proof will appear elsewhere.

RÉSUMÉ

Dans cette courte note, nous annonçons un théorème de régularité pour le flot de Kähler–Ricci sur une variété compacte de Fano (c'est-à-dire une variété kählérienne à première classe de Chern positive) et son application à l'étude du comportement limite du flot de Kähler–Ricci sur les variétés de Fano de dimension 3. Par ailleurs, nous présentons une estimation C^0 partielle du flot de Kähler–Ricci sous l'hypothèse de régularité, qui étend des travaux antérieurs concernant les métriques de Kähler–Einstein et les solitons de Kähler–Ricci régressifs. La preuve détaillée paraîtra ailleurs.

1. Introduction

Let M be a Fano n-manifold and g_0 be any Kähler metric with Kähler class $2\pi c_1(M)$. Consider the normalized Kähler–Ricci flow:

$$\frac{\partial g}{\partial t} = g - \mathrm{Ric}(g), \quad g(0) = g_0. \tag{1}$$

It was proved in [1] that (1) has a global solution $g(t)$ for $t \geqslant 0$. The main problem is to understand the limit of $g(t)$ as t tends to ∞.

By Perelman's non-collapsing result [12], there exists $\kappa = \kappa(g_0) > 0$ such that:

$$\mathrm{vol}_{g(t)}\big(B_{g(t)}(x,r)\big) \geqslant \kappa r^{2n}, \quad \forall t \geqslant 0,\ r \leqslant 1. \tag{2}$$

E-mail addresses: gtian@math.pku.edu.cn (G. Tian), zhleigo@yahoo.com.cn (Z. Zhang).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.07.005

For any sequence $t_i \to \infty$, by taking a subsequence if necessary, $(M, g(t_i))$ converge to a limiting length space (M_∞, d) in the Gromov–Hausdorff topology:

$$\big(M, g(t_i)\big) \xrightarrow{d_{GH}} (M_\infty, d). \tag{3}$$

The question is the regularity of M_∞. A desirable picture is given in the following folklore conjecture.[1]

Conjecture 1.1. *(See [18], also see [9].) $(M, g(t))$ converges (at least along a subsequence) to a shrinking Kähler–Ricci soliton with mild singularities.*

Here, "mild singularities" may be understood in two ways: (i) a singular set of codimension at least 4, and (ii) a singular set of a normal variety. By extending the partial C^0-estimate conjecture [19] to the Kähler–Ricci flow, one can show that these two approaches are actually equivalent (see Section 3 or [22]).

As pointed out by the first named author, this conjecture implies the Yau–Tian–Donaldson conjecture. The conjecture states that a Fano manifold M admits a Kähler–Einstein metrics if it is K-stable. Recently, solutions were provided for this conjecture in the case of Fano manifolds ([20], also see [6–8]).

2. Regularity of the Kähler–Ricci flow

Let $g(t)$ be a normalized Kähler–Ricci flow on a Fano manifold M and (M_∞, d) be a sequential limit as phrased in (3). The main regularity result is:

Theorem 2.1. *(See [22].) Suppose that for some uniform $p > n$ and $\Lambda < \infty$,*

$$\int_M \big|\mathrm{Ric}\big(g(t)\big)\big|^p \,\mathrm{d}v_{g(t)} \leqslant \Lambda. \tag{4}$$

Then the limit M_∞ is smooth outside a closed subset $\mathcal{S}$ of (real) codimension $\geqslant 4$ and d is induced by a smooth Kähler–Ricci soliton g_∞ on $M_\infty \backslash \mathcal{S}$. Moreover, $g(t_i)$ converges to g_∞ in the C^∞-topology outside $\mathcal{S}$.[2]

The proof of the theorem relies on Perelman's pseudolocality theorem [12] of Ricci flow and a regularity theory for manifolds with L^p bounded Ricci curvature (p bigger than half dimension) and uniformly local volume non-collapsing condition (2). This is a generalization of the regularity theories of Cheeger–Colding [2–4] and Cheeger–Colding–Tian [5]. The proof can be carried out following the lines given in these papers under the framework established by Petersen and Wei [13,14] on the geometry of manifolds with integral bounded Ricci curvature.

We shall show in [22] a uniform L^4 bound on the Ricci curvature along the Kähler–Ricci flow on any Fano manifold. The above regularity result implies:

Corollary 2.2. *(See [22].) Conjecture 1.1, i.e., the Hamilton–Tian conjecture, holds for dimension $n \leqslant 3$.*

In the case of Del-Pezzo surfaces, Conjecture 1.1 follows from [23] and [10].

3. Partial C^0 estimate of the Kähler–Ricci flow

The partial C^0 estimate of Kähler–Einstein manifolds plays the key role in Tian's program to resolve the Yau–Tian–Donaldson conjecture, see [17–20,11], for example. An extension of the partial C^0 estimate to shrinking Kähler–Ricci solitons was given in [15]. These works are based on the compactness of Cheeger–Colding–Tian [5] and its generalizations to solitons by [21]. We shall generalize these to the Kähler–Ricci flow on Fano manifolds in [22] under the regularity assumption of the limit M_∞.

Let $u(t)$ denote the Ricci potentials of the Kähler–Ricci flow $g(t)$ that satisfy:

$$\mathrm{Ric}\big(g(t)\big) + \partial\bar\partial u(t) = g(t), \qquad \int e^{-u(t)} \,\mathrm{d}v_{g(t)} = \mathrm{vol}(M). \tag{5}$$

The Hermitian metrics $\tilde g(t) = \mathrm{e}^{-\frac{1}{n}u(t)} g(t)$ have $\omega(t)$, the Kähler forms of $g(t)$, as their Chern curvature forms. Let $H(t)$ be the induced metric on K_M^{-l}, the l-th power of the anti-canonical bundle ($l \geqslant 1$). Let ∇ and $\bar\nabla$ denote the $(1,0)$ and $(0,1)$ part of the Levi-Civita connection, respectively. Then, at any time t, we have the Bochner-type formula for $\sigma \in H^0(M, K_M^{-l})$:

[1] It is often referred to as the Hamilton–Tian conjecture (see [18]).

[2] The convergence with these properties is usually referred to as the convergence in the Cheeger–Gromov topology, see [17] for instance.

$$\triangle|\nabla\sigma|^2 = |\nabla\nabla\sigma|^2 + |\bar{\nabla}\nabla\sigma|^2 - \big((n+2)l-1\big)|\nabla\sigma|^2 - \langle\partial\bar{\partial}u(\nabla\sigma,\cdot),\nabla\sigma\rangle \tag{6}$$

and the Weitzenböck-type formulas for $\xi\in C^\infty(M,T^{1,0}M\otimes K_M^{-l})$:

$$\triangle_{\bar{\partial}}\xi = \bar{\nabla}^*\bar{\nabla}\xi + (l+1)\xi - \partial\bar{\partial}u(\xi,\cdot), \tag{7}$$

$$\triangle_{\bar{\partial}}\xi = \nabla^*\nabla\xi - (n-1)l\xi, \tag{8}$$

where $\triangle_{\bar{\partial}}$ is the Hodge Laplacian of $\bar{\partial}$. Since the Sobolev constant under the Kähler–Ricci flow is uniformly bounded [24], the Moser iteration gives the gradient estimate to $\sigma\in H^0(M,K_M^{-l})$ and L^2 estimate to solutions $\bar{\partial}\vartheta=\xi\in C^\infty(M,T^{1,0}M\otimes K_M^{-l})$; compare Lemmas 4.1 and 5.4 of [20]. Perelman's C^1 estimate to $u(t)$ [16] will be used in the iteration arguments.

Now, let $\{s_{t,l,i}\}_{i=1}^{N_{t,l}}$ be an orthonormal basis of $H^0(M,K_M^{-l})$ with respect to the L^2 norm defined by $H(t)$ and Riemannian volume form, and put:

$$\rho_{t,l}(x) = \sum_{i=1}^{N_{t,l}} |s_{t,l,i}|_H^2(x), \quad \forall x\in M. \tag{9}$$

By using arguments similar to those in [11] or [20], we can prove:

Theorem 3.1 *(Partial C^0 estimate). (See [22].) If $(M,g(t_i))\xrightarrow{d_{GH}}(M_\infty,g_\infty)$ as phrased in Theorem 2.1, then the partial C^0 estimate:*

$$\inf_{t_j}\inf_{x\in M}\rho_{t_j,l}(x) > 0 \tag{10}$$

holds for a sequence of $l\to\infty$.

A direct corollary of this is to refine the regularity in Theorem 2.1.

Theorem 3.2. *(See [22].) Suppose $(M,g(t_i))\xrightarrow{d_{GH}}(M_\infty,g_\infty)$ as above. Then M_∞ is a normal projective variety and $\mathcal{S}$ is a subvariety of complex codimension at least 2.*

Finally, let us indicate how to deduce the Yau–Tian–Donaldson conjecture from the Hamilton–Tian conjecture. Suppose M is K-stable as defined in [18]. Then, under the Kähler–Ricci flow $g(t)$, we get a shrinking Kähler–Ricci soliton. From this, together with the uniqueness theorem on shrinking solitons, we can conclude that the Lie algebra of holomorphic vector fields on M_∞ is reductive. Then the K-stability implies the vanishing of Futaki invariant of M_∞, consequently, the limit (M_∞,g_∞) is Kähler–Einstein. If M_∞ is *not* biholomorphic to M, then the eigenspaces of the first eigenvalues of $-\triangle_{g(t)}+g^{i\bar{j}}(t)\partial_i u(t)\partial_{\bar{j}}$ will converge to a subspace of potential functions on M_∞ whose complex gradients are nontrivial holomorphic vector fields, cf. [25]. These vector fields induce the required degeneration of M to M_∞, with vanishing Futaki invariants. This gives a contradiction to the K-stability of M. So we have:

Theorem 3.3. *(See [22].) Suppose M is K-stable. If $(M,g(t_i))\xrightarrow{d_{GH}}(M_\infty,g_\infty)$ as phrased in Theorem 2.1, then M coincides with M_∞ and admits a Kähler–Einstein g_∞.*

In view of the regularity of low dimensional Kähler–Ricci flow in Section 2 we have:

Corollary 3.4. *(See [22].) The Yau–Tian–Donaldson conjecture holds for dimension $n\leqslant 3$.*

References

[1] H.D. Cao, Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds, Invent. Math. 81 (1985) 359–372.
[2] J. Cheeger, T.H. Colding, Lower bounds on the Ricci curvature and the almost rigidity of warped products, Ann. Math. 144 (1996) 189–237.
[3] J. Cheeger, T.H. Colding, On the structure of spaces with Ricci curvature bounded below I, J. Differential Geom. 46 (1997) 406–480.
[4] J. Cheeger, T.H. Colding, On the structure of spaces with Ricci curvature bounded below II, J. Differential Geom. 54 (2000) 13–35.
[5] J. Cheeger, T.H. Colding, G. Tian, On the singularities of spaces with bounded Ricci curvature, Geom. Funct. Anal. 12 (2002) 873–914.
[6] X.X. Chen, S. Donaldson, S. Sun, Kähler–Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities, arXiv:1211.4566.
[7] X.X. Chen, S. Donaldson, S. Sun, Kähler–Einstein metrics on Fano manifolds, II: limits with cone angle less than 2π, arXiv:1212.4714.
[8] X.X. Chen, S. Donaldson, S. Sun, Kähler–Einstein metrics on Fano manifolds, III: limits as cone angle approaches 2 and completion of the main proof, arXiv:1302.0282.
[9] X.X. Chen, G. Tian, Ricci flow on Kähler–Einstein manifolds, Duke Math. J. 131 (2006) 17–73.
[10] X.X. Chen, B. Wang, Space of Ricci flows I, Commun. Pure Appl. Math. 65 (2012) 1399–1457.
[11] S. Donaldson, S. Sun, Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry, arXiv:1206.2609v1.
[12] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159.
[13] P. Petersen, G.F. Wei, Relative volume comparison with integral curvature bounds, Geom. Funct. Anal. 7 (1997) 1031–1045.

[14] P. Petersen, G.F. Wei, Analysis and geometry on manifolds with integral Ricci curvature bounds. II, Trans. Amer. Math. Soc. 353 (2001) 457–478.
[15] D.H. Phong, J. Song, J. Sturm, Degeneration of Kähler–Ricci solitons on Fano manifolds, arXiv:1211.5849v1.
[16] N. Sesum, G. Tian, Bounding scalar curvature and diameter along the Kähler–Ricci flow (after Perelman), J. Inst. Math. Jussieu 7 (2008) 575–587.
[17] G. Tian, On Calabi's conjecture for complex surfaces with positive first Chern class, Invent. Math. 101 (1990) 101–172.
[18] G. Tian, Kähler–Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997) 1–39.
[19] G. Tian, Existence of Einstein metrics on Fano manifolds, in: Metric and Differential Geometry, Prog. Math. 297 (2012) 119–159.
[20] G. Tian, K-stability and Kähler–Einstein metrics, arXiv:1211.4669.
[21] G. Tian, Z.L. Zhang, Degeneration of Kähler–Ricci solitons, Int. Math. Res. Not. IMRN (2012) 957–985.
[22] G. Tian, Z.L. Zhang, Long-time limits of the Kähler–Ricci flow, preprint.
[23] G. Tian, X.H. Zhu, Convergence of Kähler–Ricci flow, J. Amer. Math. Soc. 20 (2007) 675–699.
[24] Q.S. Zhang, A uniform Sobolev inequality under Ricci flow, Int. Math. Res. Not. IMRN (2007) 1–12.
[25] Z.L. Zhang, Kähler–Ricci flow on Fano manifolds with vanished Futaki invariants, Math. Res. Lett. 18 (2011) 969–982.