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Group Theory/Dynamical Systems

Explicit left orders on free groups extending the lexicographic order on free monoids

Ordres à gauche explicites sur les groupes libres étendant l'ordre lexicographique sur les monoïdes libres

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Presented by the Editorial Board

To my teacher and friend, Frosina Mitruševa, on the occasion of her birthday.

ABSTRACT

For every finitely generated free group, we construct an explicit left order extending the lexicographic order on the free monoid generated by the positive letters. The order is defined by a left, free action on the orbit of 0 of a free group of piecewise linear homeomorphisms of the line. The membership in the positive cone is decidable in linear time in the length of the input word. The positive cone forms a context-free language closed under word reversal.

RÉSUMÉ

Pour tout groupe libre fini engendré, nous construisons explicitement un ordre à gauche qui étend l'ordre lexicographique sur le monoïde libre engendré par les lettres positives. Cet ordre est défini par une action à gauche, libre, sur l'orbite de 0 d'un groupe libre d'homéomorphismes de la droite linéaires par morceaux. L'appartenance au cône positif est décidable en temps linéaire par rapport à la longueur du mot. Le cône positif forme un langage non contextuel fermé par image miroir.

Version française abrégée

Un groupe G est ordonnable à gauche s'il existe un ordre total $\leqslant$ sur G compatible avec la multiplication à gauche, i.e., pour tous les éléments f, g et h de G, l'inégalité $f \leqslant g$ entraîne $hf \leqslant hg$. Il est bien connu depuis les années 1940 que le groupe libre F_k de rang k est ordonnable à gauche (et, justement, bi-ordonnable). Pourtant, la plupart des preuves précédentes sont non constructives ou trop compliquées (souvent à cause d'une volonté de généraliser ultérieurement).

Peut-être la construction la plus explicite connue à l'heure actuelle d'un ordre sur les groupes libres est donnée par l'approche de Magnus–Bergman [1], basée sur le plongement de Magnus [2] du groupe libre $F_k = F(\Sigma_k)$ dans l'anneau des séries formelles à coefficients entiers en variables non commutatives dans $\Sigma_k = \{s_1, \dots, s_k\}$. Les monômes sur Σ_k sont ordonnés par longueur lexicographique et un élément u de F_k est declaré positif si et seulement si le coefficient du monôme minimal (different de 1) dans la série formelle réprésentant u dans le plongement de Magnus est positif.

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Pour tout groupe libre finiment engendré, nous construisons explicitement un ordre à gauche qui étend l'ordre lexicographique sur le monoïde libre engendré par les lettres positives. Cet ordre est défini par une action à gauche, libre, sur l'orbite de 0 d'un groupe libre d'homéomorphismes de la droite linéaires par morceaux. L'appartenance au cône positif est décidable en temps linéaire par rapport à la longueur du mot en comptant directement les sous-mots de longueur 2 d'un certain type (voir le critère de positivité dans le théorème 0.2). Le cône positif forme un langage non contextuel fermé par image miroir.

On divise le cercle $S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\{0=1\}$ en sept arcs de même longueur, notés $1' = [0,1/7)$, $a' = [1/7,2/7)$, $b' = [2/7,3/7)$, $c' = [3/7,4/7)$, $C' = [4/7,5/7)$, $B' = [5/7,6/7)$, $A' = [6/7,1)$, et on définit trois homéomorphismes du cercle linéaires par morceaux (avec un nombre fini de singularités) conservant l'orientation a, b, et c comme dans la Fig. 1 à gauche. Les inverses de a, b et c, dénotés par A, B, et C, respectivement, sont representés dans la moitié droite de la Fig. 1. Il est évident que les inclusions (1) et (2) sont satisfaites. Par conséquent, $F = F_3 = \langle a,b,c\rangle$ est libre de rang 3. De plus, F agit librement, par une action à gauche, sur le sous-ensemble $F1'$ du cercle (notons que $F1'$ a une mesure de Lebesgue égale à 1). En particulier, il agit librement sur l'orbite de 0.

On relève les sous-intervalles du cercle S^1 à des sous-ensembles de la droite $\mathbb{R}$ suivant la projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$, et on relève les applications a, b et c à des homéomorphismes de la droite linéaires par morceaux conservant l'orientation (avec un nombre fini de singularités dans chaque sous-ensemble compact) comme dans la Fig. 2. On ne change pas de notation pour des sous-ensembles ou des homéomorphismes relevés.

Notons que le groupe F relevé est encore libre, agissant librement sur l'orbite de 0. Par conséquent, ceci définit un ordre à gauche sur F en posant $g > h$ si et seulement si $g(0) > h(0)$ (de manière équivalente, si et seulement si $h^{-1}g(0) > 0$). Un critère simple pour la positivité est obtenu en « traçant » l'action sur 0, c'est-à-dire en calculant, pour tout mot réduit u, la distance signée, avec une erreur plus petite que 1/2, entre $u(0)$ and 0.

Proposition 0.1. *Soit $w : F_3 \to \mathbb{R}$ la fonction poids définie par :*

$$w(u) = \#\{\text{des sous-mots de } u \text{ de la forme } cB,\ cA,\ \text{ou } bA\} - \#\{\text{des sous-mots de } u \text{ de la forme } Cb,\ Ca,\ \text{ou } Ba\} + \frac{1}{2}\begin{cases} 1, & \text{si } u \text{ se termine par une lettre positive, i.e., par une des lettres } a,b,c,\\ -1, & \text{si } u \text{ se termine par une lettre negative, i.e., par une des lettres } A,B,C,\\ 0, & \text{si } u \text{ est le mot trivial,}\end{cases}$$

où u est un mot réduit sur $\{a,b,c\}$. Alors $u(0) > 0$ si et seulement si $w(u) > 0$.

En suivant une construction analogue (en divisant S^1 en $2k+1$ morceaux, pour $k \geqslant 2$, et en définissant k homéomorphismes de S^1 linéaires par morceaux $s_1,\dots,s_k$, et ainsi de suite), on peut facilement établir le résultat suivant.

Théorème 0.2. *Soit $\Sigma_k = \{s_1,\dots,s_k\}$, avec $k \geqslant 2$. Pour $i = 1,\dots,k$, posons $S_i = s_i^{-1}$. On peut définir un ordre à gauche sur le groupe libre $F_k = F(\Sigma_k)$ en étendant l'ordre lexicographique sur le monoide libre Σ_k^* basé sur l'ordre $s_1 < s_2 < \cdots < s_k$ sur l'alphabet des lettres positives, comme ci-dessous. Soit $w : F_k \to \mathbb{R}$ la fonction poids définie par :*

$$w(u) = \#\{\text{des sous-mots de } u \text{ de la forme } s_jS_i,\ \text{pour } j > i\} - \#\{\text{des sous-mots de } u \text{ de la forme } S_js_i,\ \text{pour } j > i\} + \frac{1}{2}\begin{cases} 1, & \text{si } u \text{ se termine par une lettre positive } s_i,\ i = 1,\dots,k,\\ -1, & \text{si } u \text{ se termine par une lettre negative } S_i,\ i = 1,\dots,k,\\ 0, & \text{si } u \text{ est le mot trivial,}\end{cases}$$

où u est un mot réduit sur Σ_k. Alors, l'ensemble $P_k = \{u \in F_k \mid w(u) > 0\}$ est un cône positif de F_k (i.e., $u > v$ dans $F_k \Leftrightarrow w(v^{-1}u) > 0$).

1. Left orders on free groups extending the lexicographic order

A group G is left orderable if there exists a linear order $\leqslant$ on G that is compatible with the left multiplication, i.e., for all elements f, g and h in G, if $f \leqslant g$, then $hf \leqslant hg$. It has been known at least since the 1940s that the free group F_k of rank k is left orderable (and, in fact, bi-orderable, admitting an order that is compatible with both the left and the right multiplication simultaneously). In two of his papers [6,5] related to the subject, Neumann mentions that, in addition to himself, several other authors have stated this fact in published or unpublished works, including Tarski, G. Birkhoff, Shimbireva, and Iwasawa (despite his laudable effort to give credit to all, he was unaware of the simultaneous work of Vinogradov [8]). However, most of the early proofs are nonconstructive or too involved (often because of an attempt for greater generality).

Perhaps the most explicit, currently known, construction of an order on free groups is given by the Magnus–Bergman approach [1], based on the Magnus embedding [2] of the free group $F_k = F(\Sigma_k)$ into the ring of formal power series with integer coefficients in noncommuting variables from $\Sigma_k = \{s_1,\dots,s_k\}$. The monomials over Σ_k are ordered by short-lex and an element u from F_k is declared positive if and only if the coefficient in front of the smallest monomial (different from 1) in the power series representing u under the Magnus embedding is positive.

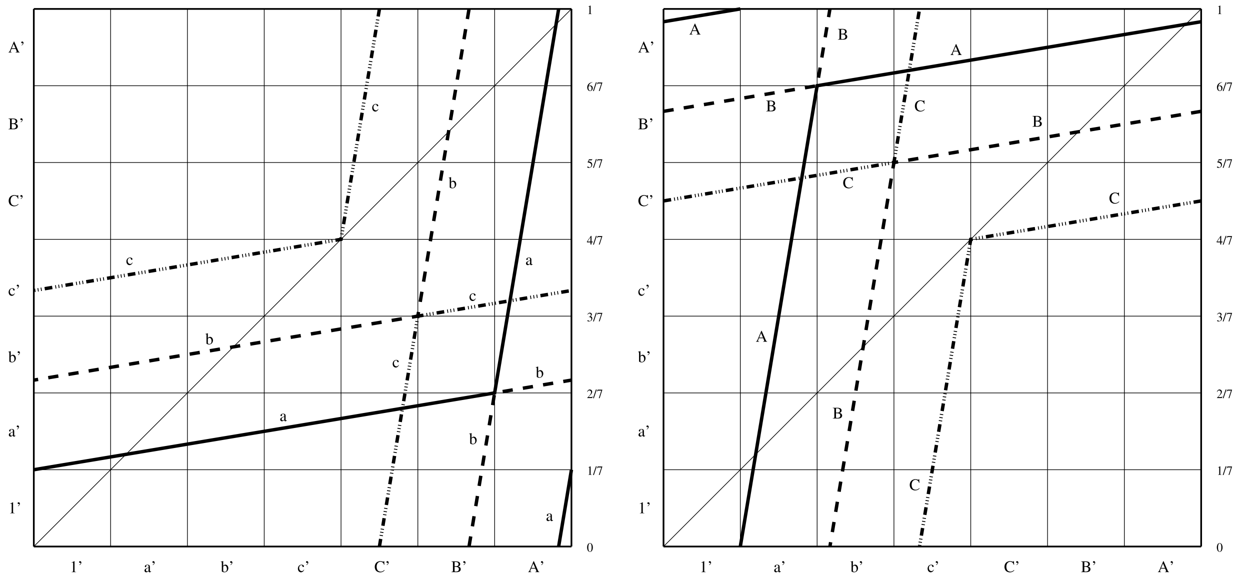

Fig. 1. Generators of F in $\mathsf{PLF}_+(S^1)$.

For every finitely generated free group, we construct an explicit left order extending the lexicographic order on the free monoid generated by the positive letters. The order is defined by a left, free action on the orbit of 0 of a free group of piecewise linear homeomorphisms of the line. The membership in the positive cone is decidable in linear time in the length of the input word by straightforward counting of subwords of length 2 of a certain type (see the positivity criterion in Theorem 1.2). The positive cone forms a context-free language closed under word reversal.

We provide details only for the case of F_3, the free group of rank 3. The general case is analogous.

Subdivide the circle $S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\{0=1\}$ into seven arcs of the same length, denoted $1' = [0,1/7)$, $a' = [1/7,2/7)$, $b' = [2/7,3/7)$, $c' = [3/7,4/7)$, $C' = [4/7,5/7)$, $B' = [5/7,6/7)$, $A' = [6/7,1)$, and define three piecewise linear, orientation preserving homeomorphisms a, b, and c of the circle (with finitely many breaks) as in the left half of Fig. 1. The inverses of a, b and c are denoted by A, B, and C, respectively, and are shown in the right half of Fig. 1. It is evident that:

$$a\big(S^1 \setminus A'\big) \subseteq a', \qquad b\big(S^1 \setminus B'\big) \subseteq b', \qquad c\big(S^1 \setminus C'\big) \subseteq c', \tag{1}$$

$$A\big(S^1 \setminus a'\big) \subseteq A', \qquad B\big(S^1 \setminus b'\big) \subseteq B', \qquad C\big(S^1 \setminus c'\big) \subseteq C'. \tag{2}$$

Therefore, $F = F_3 = \langle a,b,c\rangle$ is free of rank 3. Moreover, F acts freely, through a left action, on the subset $F1'$ of the circle (note that $F1'$ has Lebesgue measure 1). In particular, it acts freely on the orbit of 0.

We lift the subintervals from the circle S^1 to subsets of the line $\mathbb{R}$ along the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$, and we lift the maps a, b and c to piecewise linear, orientation preserving the homeomorphisms of the line (with finitely many breaks on every compact subset) as in Fig. 2. We do not change the notation for the lifted subsets or homeomorphisms.

Note that the lifted group F is still free, acting freely on the orbit of 0. Therefore, a left order is defined on F by declaring $g > h$ if and only if $g(0) > h(0)$ (equivalently, if and only if $h^{-1}g(0) > 0$). A simple criterion for positivity is obtained by “tracing” the action on 0, i.e., by calculating, for reduced group words u, the signed distance, within error smaller than $1/2$, between $u(0)$ and 0. Of course, since the action is given explicitly and the orbits of rational points are rational, $u(0)$ can be calculated exactly, but that takes longer, obscures the features of the order, and is not necessary.

Proposition 1.1. *Define a weight function $w : F_3 \to \mathbb{R}$ by*

$$w(u) = \#\{\textit{of subwords of } u \textit{ of the form } cB,\ cA,\ \textit{or } bA\} - \#\{\textit{of subwords of } u \textit{ of the form } Cb,\ Ca,\ \textit{or } Ba\}$$
$$+\frac{1}{2}\begin{cases} 1, & \textit{if } u \textit{ ends in a positive letter, i.e., one of the letters } a,b,c,\\ -1, & \textit{if } u \textit{ ends in a negative letter, i.e., one of the letters } A,B,C,\\ 0, & \textit{if } u \textit{ is the trivial word},\end{cases}$$

where u is a reduced group word over $\{a,b,c\}$. Then

$$u(0) > 0 \quad \textit{if and only if} \quad w(u) > 0.$$

Indeed, for any nontrivial word u, the point $u(0)$ is in exactly one of the intervals

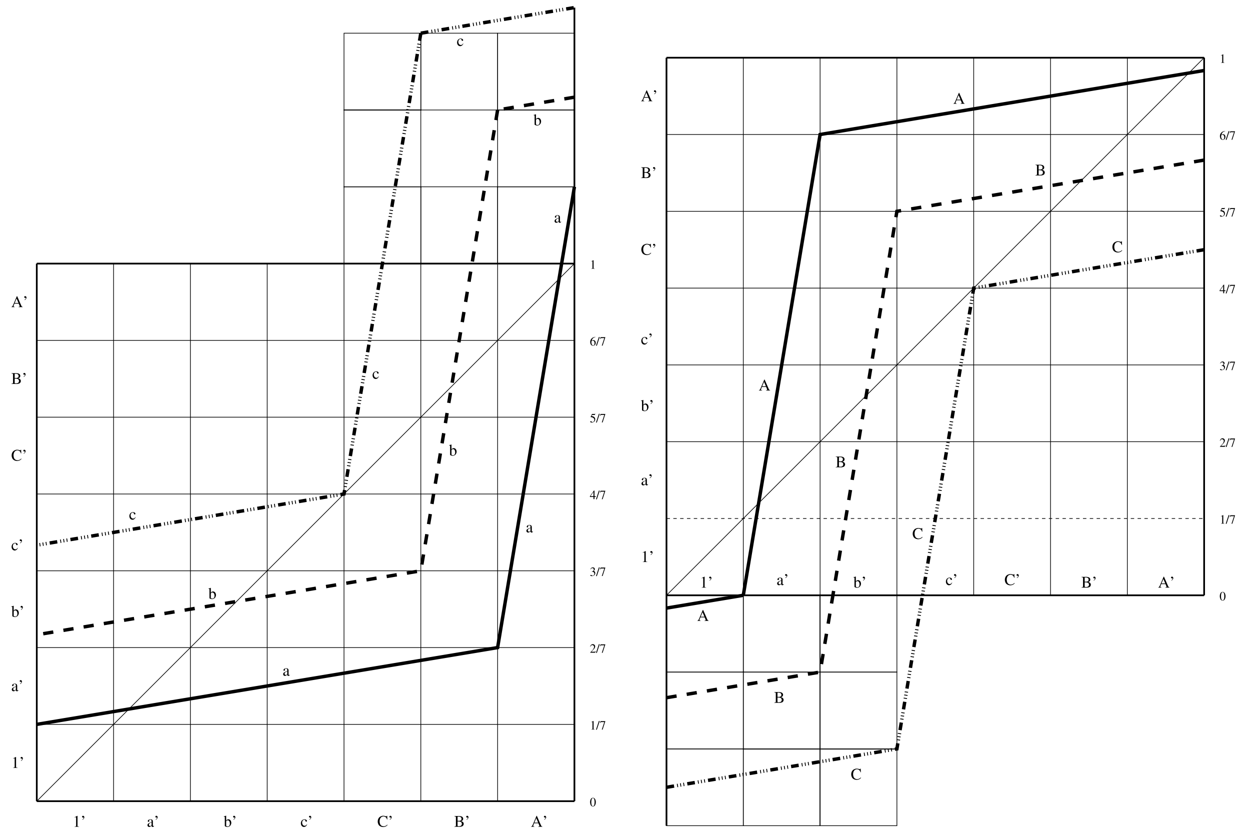

Fig. 2. Generators of F in $\mathsf{PLF}_+(\mathbb{R})$.

$$\ldots, (-2, -1), (-1, 0), (0, 1), (1, 2), \ldots.$$

We claim that $w(u)$ represents the midpoint of the interval to which $u(0)$ belongs. By definition, the action of F is a left action, hence the last letter of u acts first on 0 and pushes it into the interval $(0, 1)$ if the letter is positive, or to $(-1, 0)$ if the letter is negative, and the part of the formula for w related to the last letter of u records this as $1/2$ or $-1/2$. From here on, we trace the action of u (from right to left) on the obtained point, but only record jumps into the next interval up (by adding 1 in our weight) and jumps into the next interval down (by subtracting 1 in our weight). Negative letters never produce jumps up (see the right half of Fig. 2) and positive letters never produce jumps down (see the left half of Fig. 2). By examining the action of the positive letters in the left half of Fig. 2, we see that jumps up occur exactly when the letter c is applied to a point in the regions B' and A', which is precisely when a subword of the form cB or cA occurs in u, or when the letter b is applied to a point in the region A', which is precisely when a subword of the form bA occurs in u. We exclude the possibility of applying the letter c to the region C', b to the region B', and a to the region A', because u is a reduced word. Similarly, by examining the action of the negative letters in the right half of Fig. 2, we see that jumps down occur exactly when the letter C is applied to a point in the regions b' and a', which is precisely when a subword of the form Cb or Ca occurs in u, or when the letter B is applied to a point in the region a', which is precisely when a subword of the form Ba occurs in u. We exclude the possibility of applying the letter C to the region c', B to the region b', and A to the region a', because u is a reduced word. We also exclude the possibility of applying any of the negative letters to the region $1'$, since the point 0 will never return to the region $1'$ under the action of a nontrivial word u, after, in the very first step, the last letter of u moves it from there.

We claim that the given order extends the usual lexicographic order on the free monoid $M_3 = \{a, b, c\}^*$ based on $a < b < c$. All we need to verify is that, for all words v_1, v_2, v_3 in M_3, $e < av_1 < bv_2 < cv_3$, i.e., we need to verify that $w(av_1)$, $w(v_1^{-1}Abv_2)$, $w(v_2^{-1}Bcv_3) > 0$. Since the weight of each of these words is $1/2$, the claim is correct.

In fact, it is possible to see that the order, restricted to words in $M_3 = \{a, b, c\}^*$ is lexicographic just by looking at the left half of Fig. 2. Namely, for u in M_3, $u(0)$ is trapped in the interval $[0, 4/7)$, and, for all words v_1, v_2, v_3 in M_3, we have $0 < av_1(0) < bv_2(0) < cv_3(0)$, since, on the interval $[0, 4/7)$ the entire graph of the function a is above 0 and below the minimum of the function b, and the entire graph of the function b is below the minimum of the function c.

Following an analogous construction (subdividing S^1 into $2k + 1$ pieces, for $k \geqslant 2$, defining k piecewise linear homeomorphisms $s_1, \ldots, s_k$ of S^1, and so on), we may easily establish the following result.

Theorem 1.2. *Let $\Sigma_k = \{s_1, \ldots, s_k\}$, for some $k \geqslant 2$. For $i = 1, \ldots, k$, denote $S_i = s_i^{-1}$. A left order on the free group $F_k = F(\Sigma_k)$ extending the lexicographic order on the free monoid Σ_k^* based on the order $s_1 < s_2 < \cdots < s_k$ on the alphabet of positive letters may be defined as follows. Define a weight function $w : F_k \to \mathbb{R}$ by:*

$$w(u) = \#\{\text{of subwords of } u \text{ of the form } s_j S_i,\ \text{for } j > i\} - \#\{\text{of subwords of } u \text{ of the form } S_j s_i,\ \text{for } j > i\} + \frac{1}{2}\begin{cases} 1, & \text{if } u \text{ ends in any positive letter } s_i,\ i = 1, \ldots, k, \\ -1, & \text{if } u \text{ ends in any negative letter } S_i,\ i = 1, \ldots, k, \\ 0, & \text{if } u \text{ is the trivial word}, \end{cases}$$

where u is a reduced group word over Σ_k, and declare that the set:

$$P_k = \{u \in F_k \mid w(u) > 0\}$$

is the positive cone of F_k (i.e., $u > v$ in $F_k \Leftrightarrow w(v^{-1}u) > 0$).

The construction of the k homeomorphisms $s_1, \ldots, s_k$ of S^1 that generate a free group acting freely on the orbit of 0 and that can be used to prove Theorem 1.2 relies on the general approach for producing quasi-Schottky groups in [7]. They may be defined as follows. For $i = 1, \ldots, k$, let $s_i : S^1 \to S^1$ be the homeomorphism of the circle given by:

$$s_i = s_0\left(x + \frac{i-1}{2k+1}\right) + \frac{i}{2k+1}, \quad \text{where } s_0(x) = \begin{cases} \frac{1}{2k}x, & 0 \leqslant x < \frac{2k}{2k+1}, \\ 2kx - (2k-1), & \frac{2k}{2k+1} \leqslant x < 1. \end{cases}$$

The membership problem for the positive cone P_k is rather easy and can be solved in linear time in the length of the input word. If we count the relevant subwords as we read, we can calculate the weight and tell if a word is in the positive cone by the time we finish reading the word.

It is known that the positive cone of a left order of a free group cannot be finitely generated as a monoid (this can be deduced from the work of McCleary [3], but was the first explicit proof is given by Navas [4]), which implies that it cannot be a regular language of the form Y^* for some finite set Y of group words. On the other hand, it is apparent that the positive cone P_k is a context free language over $\Sigma_k^{\pm}$ (indeed, a push down automaton with a single stack can easily establish if the number of subwords of the form $s_j S_i$, with $j > i$ in a given word u is greater than, smaller than, or equal to the number of subwords of the form $S_j s_i$, with $j > i$, and can take into account the last letter of u in case of a tie).

Another interesting feature of the positive cone P_k is that it is closed under word reversal. Indeed, for a given word u, with $w(u) > 0$, we have $w((u^{\mathrm{R}})^{-1}) = -w(u) < 0$, where u^{R} denotes the word reversal of u, since the transformation $(u^{\mathrm{R}})^{-1}$ just exchanges the positive and negative letters (while keeping them in the same order as in u), and the effect of this on the weight is to exactly exchange all positive and all negative contributions. Since $w((u^{\mathrm{R}})^{-1}) < 0$, we must have $w(u^{\mathrm{R}}) > 0$.

The left order provided in Theorem 1.2 is not two-sided (unlike the Magnus–Bergman order, which is). In fact, it is clear that no order extending the lexicographic order can be two-sided (this is because $s_1 < s_1 s_1$, but $s_1 s_2 > s_1 s_1 s_2$).

Variations of the construction presented above lead to other explicitly stated orders on free groups, not necessarily extending the lexicographic order (for instance, in case of F_2, add 1 to the weight for every subword of the form ab or aB, subtract 1 for every subword of the form BA or Ba, and add plus or minus 1/2; depending on the last letter). In fact, Cantor sets of left orders on F_k may be constructed by stringing together variations of the above construction (subsequent work).

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