C. R. Acad. Sci. Paris, Ser. I 351 (2013) 433–436

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical Analysis

A Beurling type theorem in weighted Bergman spaces

Un théorème de type Beurling dans des espaces de Bergman pondérés

Joseph A. Ball [a], Vladimir Bolotnikov [b]

[a] *Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA*
[b] *Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA*

ARTICLE INFO

Article history:
Received 25 March 2013
Accepted after revision 19 June 2013
Available online 9 July 2013

Presented by Gilles Pisier

ABSTRACT

For the vector-valued Hardy space $H^2(\mathcal{U})$ and the standard weighted Bergman space $\mathcal{A}_n(\mathcal{Y})$ with coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, we single out a class of contractive multipliers from $H^2(\mathcal{U})$ to $\mathcal{A}_n(\mathcal{Y})$ which we call *partially isometric multipliers*. We then show that a closed subspace $\mathcal{M} \subset \mathcal{A}_n(\mathcal{Y})$ is invariant under the shift operator $S_n : f(z) \mapsto zf(z)$ if and only if $\mathcal{M} = \Phi \cdot H^2(\mathcal{U})$ for some partially isometric multiplier Φ from $H^2(\mathcal{U})$ to $\mathcal{A}_n(\mathcal{Y})$.

RÉSUMÉ

Soit $H^2(\mathcal{U})$ l'espace de Hardy aux valeurs vectorielles et soit $\mathcal{A}_n(\mathcal{Y})$ l'espace de Bergman aux valeurs vectorielles et au poids $(1-|z|^2)^{n-2}$, où les espaces des coefficients $\mathcal{U}$ et $\mathcal{Y}$ sont des espaces de Hilbert. Nous considérons une classe de multiplicateurs contractifs de $H^2(\mathcal{U})$ dans $\mathcal{A}_n(\mathcal{Y})$, que nous appelons *multiplicateurs isométriques partiels*. Nous montrons qu'un sous-espace $\mathcal{M} \subset \mathcal{A}_n(\mathcal{Y})$ qui est invariant pour l'operateur $S_n : f(z) \mapsto zf(z)$ est inclus isometriquement dans $\mathcal{A}_n(\mathcal{Y})$ si et seulement si $\mathcal{M} = \Phi \cdot H^2(\mathcal{U})$ pour un multiplicateur isométrique partiel Φ de $H^2(\mathcal{U})$ dans $\mathcal{A}_n(\mathcal{Y})$.

1. Introduction

For a Hilbert space $\mathcal{Y}$, let $H^2(\mathcal{Y})$ be the standard Hardy space of square-summable $\mathcal{Y}$-valued functions on the open unit disk $\mathbb{D}$. The shift operator $S_1 : f(z) \to zf(z)$ is an isometry on $H^2(\mathcal{Y})$ and therefore it possesses the *wandering subspace property*: any S_1-invariant subspace $\mathcal{M} \subset H^2(\mathcal{Y})$ is generated by the *wandering subspace* $\mathcal{E} = \mathcal{M} \ominus S_1\mathcal{M}$ and moreover $S_1^k\mathcal{E} \perp S_1^\ell\mathcal{E}$ for all nonnegative $k \neq \ell$. Furthermore, any such wandering subspace has the form $\mathcal{E} = \Theta \cdot \mathcal{U}$ for some inner function $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and an appropriate coefficient Hilbert space $\mathcal{U}$, which in turn leads to the representations:

$$\mathcal{M} = \bigoplus_{k\geqslant 0}\left(S_1^k\mathcal{M} \ominus S_1^{k+1}\mathcal{M}\right) = \bigoplus_{k\geqslant 0} S_1^k\mathcal{E} = \bigoplus_{k\geqslant 0} S_1^k\Theta \cdot \mathcal{U} = \Theta \cdot H^2(\mathcal{U}) \tag{1}$$

for an S_1-invariant subspace $\mathcal{M} \subset H^2(\mathcal{Y})$. These representations display the vector-valued version of the classical Beurling theorem [4] based on Halmos' wandering subspace approach [5]. We denote by $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ the space of bounded linear

E-mail addresses: joball@math.vt.edu (J.A. Ball), vladi@math.wm.edu (V. Bolotnikov).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.06.004

Hilbert space operators from $\mathcal{U}$ to $\mathcal{Y}$ and we recall that a function $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is *inner* if the multiplication operator $M_\Theta : f(z) \mapsto \Theta(z) \cdot f(z)$ is an isometry from $H^2(\mathcal{U})$ to $H^2(\mathcal{Y})$.

Beurling type characterizations of shift-invariant subspaces of the Bergman space are due to Aleman, Richter and Sundberg [2]. For an integer $n \geqslant 2$, let $\mathcal{A}_n(\mathcal{Y})$ be the space of $\mathcal{Y}$-valued functions analytic on $\mathbb{D}$ and square-integrable over $\mathbb{D}$ with respect to the weight $(1 - |z|^2)^{n-2}$. The space $\mathcal{A}_n(\mathcal{Y})$ can be alternatively characterized as:

$$\mathcal{A}_n(\mathcal{Y}) = \left\{ f(z) = \sum_{j\geqslant 0} f_j z^j : \ \|f\|^2_{\mathcal{A}_n(\mathcal{Y})} := \sum_{j\geqslant 0} \mu_{n,j} \cdot \|f_j\|^2_{\mathcal{Y}} < \infty \right\}, \quad \mu_{n,j} = \frac{j!(n-1)!}{(j+n-1)!}, \tag{2}$$

or just as the reproducing kernel Hilbert space with reproducing kernel $k_{\mathcal{A}_n(\mathcal{Y})}(z, \zeta) = (1 - z\bar{\zeta})^{-n} I_{\mathcal{Y}}$. If we interpret the binomial coefficient $\mu_{n,j}$ in (2) to have value 1 when $n = 1$, the two latter characterizations identify $\mathcal{A}_1(\mathcal{Y})$ with the Hardy space $H^2(\mathcal{Y})$. We let $S_n : f(z) \mapsto zf(z)$ denote the shift operator on $\mathcal{A}_n(\mathcal{Y})$, and we recall that a function $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is called $\mathcal{A}_n(\mathcal{Y})$*-inner* if $\|\Theta u\|_{\mathcal{A}_n(\mathcal{Y})} = \|u\|_{\mathcal{U}}$ for all $u \in \mathcal{U}$ and if $\Theta u \perp S_n^k \Theta v$ for all $u, v \in \mathcal{U}$ and all integers $k \geqslant 1$; see [2].

As was shown in [2], the Bergman shift S_2 possesses the wandering subspace property (i.e., any S_2-invariant subspace $\mathcal{M}$ has the form $\mathcal{M} = \bigvee_{k\geqslant 0} S_2^k \mathcal{E}$ where $\mathcal{E} := \mathcal{M} \ominus S_2\mathcal{M}$ and where $\bigvee$ means the *closed linear span*) and moreover, if $\mathcal{M}$ is an S_2-invariant subspace of $\mathcal{A}_2(\mathcal{Y})$, then the wandering subspace $\mathcal{E} = \mathcal{M} \ominus S_2\mathcal{M}$ has the form $\mathcal{E} = \Theta_2 \cdot \mathcal{U}$ for some $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued $\mathcal{A}_2(\mathcal{Y})$-inner function Θ_2. Hence,

$$\mathcal{M} = \bigvee_{k\geqslant 0} S_2^k \mathcal{E} \quad \text{and} \quad \mathcal{E} := \mathcal{M} \ominus S_2\mathcal{M} = \Theta_2 \cdot \mathcal{U} \perp S_2^k \mathcal{E} \quad \text{for all } k \geqslant 1. \tag{3}$$

Representations $\mathcal{M} \ominus S_n\mathcal{M} = \Theta_n \mathcal{U}$ for wandering subspaces in $\mathcal{A}_n(\mathcal{Y})$ hold true for $n \geqslant 2$. It is known that S_n possesses the wandering subspace property for $n = 3$ (due to Shimorin [9]) but that S_n fails to have the wandering subspace property for $n \geqslant 4$ (see [6]).

Another representation for S_n-invariant subspaces $\mathcal{M} \subset \mathcal{A}_n(\mathcal{Y})$ is based on the observation that for any such $\mathcal{M}$, the subspace $S_n^k \mathcal{M} \ominus S_n^{k+1}\mathcal{M}$ can be always represented as $S_n^k \Theta_{n,k} \mathcal{U}_k$ for an appropriate Hilbert spaces $\mathcal{U}_k$ and an $\mathcal{A}_n(\mathcal{Y})$-inner function $z^k \Theta_{n,k}(z)$. This observation leads to the orthogonal representation:

$$\mathcal{M} = \bigoplus_{k\geqslant 0} \left(S_n^k \mathcal{M} \ominus S_n^{k+1}\mathcal{M}\right) = \bigoplus_{k\geqslant 0} S_n^k \Theta_{n,k} \mathcal{U}_k \tag{4}$$

of $\mathcal{M}$ in terms of a Bergman-inner family $\{\Theta_{n,k}\}_{k\geqslant 0}$. We refer to [3] for precise definitions and proofs only noting here that if $n = 1$, then $\Theta_{n,k}$ and $\mathcal{U}_k$ do not depend on k and representation (4) amounts to (1). So, both representations (3) and (4) originate to the Halmos wandering subspace representation (1).

In this note, we present another representation for S_n-invariant subspaces of $\mathcal{A}_n(\mathcal{Y})$ which can be traced to the de Branges–Rovnyak approach to the Beurling theorem in the classical Hardy space case.

2. Partially isometric multipliers and the main result

A function $\Phi : \mathbb{D} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is called a *contractive multiplier* from $H^2(\mathcal{U})$ to $\mathcal{A}_n(\mathcal{Y})$, denoted as $\Phi \in \mathcal{S}_n(\mathcal{U}, \mathcal{Y})$, if the multiplication operator $M_\Phi : f \to \Phi f$ is a contraction from $H^2(\mathcal{U})$ to $\mathcal{A}_n(\mathcal{Y})$. The latter is equivalent to the kernel:

$$K_\Phi(z, \zeta) := k_{\mathcal{A}_n(\mathcal{Y})}(z, \zeta) - \Phi(z) k_{H^2(\mathcal{U})}(z, \zeta) \Phi(\zeta)^* = \frac{I_{\mathcal{Y}}}{(1 - z\bar{\zeta})^n} - \frac{\Phi(z)\Phi(\zeta)^*}{1 - z\bar{\zeta}} \tag{5}$$

being positive on $\mathbb{D} \times \mathbb{D}$. With any $\Phi \in \mathcal{S}_n(\mathcal{U}, \mathcal{Y})$ we therefore can associate the reproducing kernel Hilbert space $\mathcal{H}(K_\Phi)$ with reproducing kernel K_Φ. It is readily seen from (5) that $\mathcal{H}(K_\Phi)$ is contractively included in $\mathcal{A}_n(\mathcal{Y})$ for any $\Phi \in \mathcal{S}_n(\mathcal{U}, \mathcal{Y})$. Recall that any $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued $\mathcal{A}_n(\mathcal{Y})$-inner function is in $\mathcal{S}_n(\mathcal{U}, \mathcal{Y})$ (see [8,9]). In fact, it can be shown that a contractive multiplier $\Phi \in \mathcal{S}_n(\mathcal{U}, \mathcal{Y})$ is $\mathcal{A}_n(\mathcal{Y})$-inner if and only if $\|\Theta u\|_{\mathcal{A}_n(\mathcal{Y})} = \|u\|_{\mathcal{U}}$ for all $u \in \mathcal{U}$. We now introduce another subclass of $\mathcal{S}_n(\mathcal{U}, \mathcal{Y})$.

Definition 2.1. A contractive multiplier $\Phi \in \mathcal{S}_n(\mathcal{U}, \mathcal{Y})$ will be called a *partially isometric* (**p.i.**) multiplier if the associated space $\mathcal{H}(K_\Phi)$ is *isometrically* included into $\mathcal{A}_n(\mathcal{Y})$, or equivalently (upon passing to orthogonal complements in $\mathcal{A}_n(\mathcal{Y})$), if M_Φ is a partial isometry from $H^2(\mathcal{U})$ into $\mathcal{A}_n(\mathcal{Y})$.

The next theorem characterizes closed S_n-invariant subspace of $\mathcal{A}_n(\mathcal{Y})$ in terms of **p.i.** multipliers.

Theorem 2.2. *Let n be an integer with $n \geqslant 2$. Then $\mathcal{M}$ is a closed S_n-invariant subspace of $\mathcal{A}_n(\mathcal{Y})$ if and only if there is a Hilbert space $\mathcal{U}$ and a* **p.i.** *multiplier $\Phi \in \mathcal{S}_n(\mathcal{U}, \mathcal{Y})$ such that $\mathcal{M} = \Phi \cdot H^2(\mathcal{U})$.*

Proof. The "if" part is straightforward: if $\Phi \in \mathcal{S}_n(\mathcal{U},\mathcal{Y})$ is a **p.i.** multiplier, then the range space $\operatorname{Ran} M_\Phi = \Phi \cdot H^2(\mathcal{U})$ is a closed subspace of $\mathcal{A}_n(\mathcal{Y})$ (by Definition 2.1); therefore, and since $H^2(\mathcal{U})$ is S_1-invariant, the space $\operatorname{Ran} M_\Phi$ is S_n-invariant, we now sketch the proof of the "only if" part.

Recall that an operator $A \in \mathcal{L}(X)$ is *strongly stable* if A^k converges to zero in the strong operator topology. If $\mathcal{M}$ is a closed S_n-invariant subspace of $\mathcal{A}_n(\mathcal{Y})$, then its orthogonal complement $\mathcal{M}^\perp = \mathcal{A}_n(\mathcal{Y}) \ominus \mathcal{M}$ is a closed S_n^*-invariant subspace of $\mathcal{A}_n(\mathcal{Y})$. Then there exists a Hilbert space $\mathcal{X}$, a strongly stable contraction A on $\mathcal{X}$ (in fact, an n-hypercontraction; see [1] for the definition) and an operator $C \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ such that:

$$C^*C = \sum_{j=0}^{n} (-1)^j \binom{n}{j} A^{*j}A^j, \tag{6}$$

and so that $\mathcal{M}^\perp$ has the representation $\mathcal{M}^\perp = \{C(I-zA)^{-n}x\colon\ x \in \mathcal{X}\}$. One such choice is $\mathcal{X} = \mathcal{M}^\perp$ with $C : f \mapsto f(0)$ and $A = S_n^*|_{\mathcal{M}^\perp}$; see [3, Theorem 5.3]. Furthermore, $\mathcal{M}^\perp$ can be identified as a reproducing kernel Hilbert space with reproducing kernel $K_{\mathcal{M}^\perp}(z,\zeta) = C(I-zA)^{-n}(I-\bar{\zeta}A^*)^{-n}C^*$. Therefore, the reproducing kernel for $\mathcal{M} = (\mathcal{M}^\perp)^\perp$ is given by:

$$K_{\mathcal{M}}(z,\zeta) = k_{\mathcal{A}_n(\mathcal{Y})}(z,\zeta) - K_{\mathcal{M}^\perp}(z,\zeta) = (1-z\bar{\zeta})^{-n} I_{\mathcal{Y}} - C(I-zA)^{-n}\big(I-\bar{\zeta}A^*\big)^{-n}C^*.$$

By [3, Proposition 4.7], for a strongly stable contraction $A \in \mathcal{L}(\mathcal{X})$ and $C \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ subject to equality (6), the following series converges in the strong operator topology to the identity operator:

$$\sum_{j=0}^{\infty} \mu_{n,j}^{-1} A^{*j}C^*CA^j = I_{\mathcal{X}}. \tag{7}$$

We next introduce the operators $\mathcal{O}_{C,A} : \mathcal{X} \to \ell^2(\mathcal{Y})$ and $D : \mathcal{U} \to \ell^2(\mathcal{Y})$ of the form:

$$\mathcal{O}_{C,A} = \begin{bmatrix} \mu_{n-1,0}^{-\frac{1}{2}} C \\ \mu_{n-1,1}^{-\frac{1}{2}} CA \\ \mu_{n-1,2}^{-\frac{1}{2}} CA^2 \\ \vdots \end{bmatrix}, \qquad D = \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ \vdots \end{bmatrix}, \tag{8}$$

where $\mu_{n-1,j}$ are the binomial weights defined as in (2). The first operator is completely determined from the pair (C,A) and is a contraction since by (7) and the binomial identity $\mu_{n-1,j}^{-1} + \mu_{n,j-1}^{-1} = \mu_{n,j}^{-1}$,

$$\mathcal{O}_{C,A}^*\mathcal{O}_{C,A} + A^*A = \sum_{j=0}^{\infty} \mu_{n-1,j}^{-1} A^{*j}C^*CA^j + \sum_{j=0}^{\infty} \mu_{n,j}^{-1} A^{*j+!}C^*CA^{j+1} = \sum_{j=0}^{\infty} \mu_{n,j}^{-1} A^{*j}C^*CA^j = I_{\mathcal{X}}.$$

The second operator in (8) will be chosen along with another operator $B \in \mathcal{L}(\mathcal{U},\mathcal{X})$ to solve the Cholesky factorization problem:

$$\begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}} & 0 \\ 0 & I_{\ell^2(\mathcal{Y})} \end{bmatrix} - \begin{bmatrix} A \\ \mathcal{O}_{C,A} \end{bmatrix} \begin{bmatrix} A^* & \mathcal{O}_{C,A}^* \end{bmatrix}. \tag{9}$$

With all these operators in hand we define the function $\Phi(z) = \sum_{j=0}^{\infty} \mu_{n-1,j}^{-\frac{1}{2}} D_j z^j + zC(I-zA)^{-n}B$, and a calculation based on (9) shows that Φ satisfies the identity:

$$\frac{\Phi(z)\Phi(\zeta)^*}{1-z\bar{\zeta}} = (1-z\bar{\zeta})^{-n} I_{\mathcal{Y}} - C(I-zA)^{-n}\big(I-\bar{\zeta}A^*\big)^{-n}C^* = K_{\mathcal{M}}(z,\zeta), \tag{10}$$

from which the representation $\mathcal{M} = \Phi \cdot H^2(\mathcal{U})$ follows along with the fact that Φ is a **p.i.** multiplier. □

Remark 2.3. Note that Theorem 2.2 follows from the factorization (10) of $K_{\mathcal{M}}(z,\zeta)$; such a factorization appears in a more implicit form in [3].

Remark 2.4. By using the same arguments as in [7, Theorem 10], one can show that the **p.i.** multiplier representing an S_n-invariant subspace $\mathcal{M}$ is essentially unique: if $\Phi \in \mathcal{S}_n(\mathcal{U},\mathcal{Y})$ and $\widetilde{\Phi} \in \mathcal{S}_n(\widetilde{\mathcal{U}},\mathcal{Y})$ are two partially isometric multipliers such that $\Phi \cdot H^2(\mathcal{U}) = \widetilde{\Phi} \cdot H^2(\widetilde{\mathcal{U}})$, then there exists a partial isometry $V : \widetilde{\mathcal{U}} \to \mathcal{U}$ such that $\Phi(z) = \widetilde{\Phi}(z)V$.

Remark 2.5. If $\Phi \in \mathcal{S}_n(\mathcal{U}, \mathcal{Y})$ is a **p.i.** multiplier, then the space $\mathcal{H}(K_\Phi)$ with reproducing kernel (5) is isometrically equal to $(\Phi \cdot H^2(\mathcal{U}))^\perp$ and hence is invariant under the backward shift S_n^*. In contrast to the classical case $n = 1$, this backward-shift invariance property fails for general contractive multipliers. For example, the function $\Phi(z) \equiv 1$ belongs to $\mathcal{S}_2(\mathbb{C}, \mathbb{C})$ and the corresponding kernel (5) equals $\frac{z\bar{\zeta}}{(1-z\bar{\zeta})^2}$. Therefore any function in the space $\mathcal{H}(K_\Phi)$ vanishes at the origin and hence $\mathcal{H}(K_\Phi)$ is not S_n^*-invariant. An open question is to characterize those contractive multipliers $\Phi \in \mathcal{S}_n(\mathcal{U}, \mathcal{Y})$ for which the associated space $\mathcal{H}(K_\Phi)$ is invariant under S_n^*.

References

[1] J. Agler, Hypercontractions and subnormality, J. Oper. Theory 13 (2) (1985) 203–217.
[2] A. Aleman, S. Richter, C. Sundberg, Beurling's theorem for the Bergman space, Acta Math. 177 (1996) 275–310.
[3] J.A. Ball, V. Bolotnikov, Weighted Bergman spaces: shift-invariant subspaces and input/state/output linear systems, Integral Equ. Oper. Theory 76 (3) (2013) 301–356.
[4] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math. 81 (1949) 239–255.
[5] P.R. Halmos, Shifts on Hilbert spaces, J. Reine Angew. Math. 208 (1961) 102–112.
[6] H. Hedenmalm, Y.G. Perdomo, Mean value surfaces with prescribed curvature form, J. Math. Pures Appl. 83 (2004) 1075–1107.
[7] S. McCullough, T.T. Trent, Invariant subspaces and Nevanlinna–Pick kernels, J. Funct. Anal. 178 (2000) 226–249.
[8] A. Olofsson, A characteristic operator function for the class of n-hypercontractions, J. Funct. Anal. 236 (2006) 517–545.
[9] S. Shimorin, On Beurling-type theorems in weighted ℓ^2 and Bergman spaces, Proc. Amer. Math. Soc. 131 (6) (2003) 1777–1787.