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On an extension of a bilinear functional on $\mathrm{L}^p(\mathbf{R}^d) \otimes E$ to a Bochner space with an application to velocity averaging

Sur une extension d'une fonctionnelle bilinéaire sur $\mathrm{L}^p(\mathbf{R}^d) \otimes E$ aux espaces du Bochner avec une application sur la moyennisation en vitesse

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ABSTRACT

We examine necessary and sufficient conditions under which a continuous bilinear functional B on $\mathrm{L}^p(\mathbf{R}^d) \otimes E$, $p > 1$, E being a separable Banach space, can be continuously extended to a linear functional on $\mathrm{L}^p(\mathbf{R}^d; E)$. The extension enables a generalisation of the H-distribution concept, allowing us to obtain a (heterogeneous) velocity averaging result in the L^p framework for any $p > 1$.

RÉSUMÉ

Nous examinons les conditions nécessaires et suffisantes pour qu'une fonctionelle bilinéaire continue sur $\mathrm{L}^p(\mathbf{R}^d) \otimes E$, $p > 1$, E étant un espace de Banach séparable, peut être étendue à une fonctionnelle linaire sur $\mathrm{L}^p(\mathbf{R}^d; E)$. L'extension permet une généralisation de l'H-distribution, qui fournit l'amélioration d'un résultat de moyennisation en vitesse (hétèrogène) sur le cadre L^p pour tout $p > 1$.

1. Introduction

The question of the extension of a bilinear functional from a tensor product $E \otimes F$ of two Banach spaces to a more complicated structure is classical in functional analysis. Probably the best-known example is the Schwartz kernel theorem, stating that a continuous bilinear functional B on $\mathrm{C}(X) \otimes \mathrm{C}(Y)$, $X \subset \mathbf{R}^d$, $Y \subset \mathbf{R}^m$, can be continuously extended to a distribution $B \in \mathcal{D}'(X \times Y)$.

Among many notable applications of the Schwartz kernel theorem, we mention H-measures [8,15] and their variants ([2,11] and references therein). Roughly speaking, all of them measure the loss of strong precompactness of a sequence (u_n) converging weakly to zero in $\mathrm{L}^p(\mathbf{R}^d)$ for an appropriate $p \geqslant 2$.

An H-measure is initially defined as a bilinear functional on $\mathrm{C}_0(\mathbf{R}^d) \otimes \mathrm{C}(\mathrm{S}^{d-1})$ where S^{d-1} is the sphere in $\mathbf{R}^d$. Thus, according to the Schwartz kernel theorem, it is a distribution from $\mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$. Since it can be proved that it is positively

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definite, according to the Schwartz lemma on non-negative distributions, one can also state that it is a Radon measure on $\mathbf{R}^d \times \mathrm{S}^{d-1}$.

In an extension (called H-distributions [3]) of the H-measure concept to $\mathrm{L}^p(\mathbf{R}^d)$ sequences, $p > 1$, the lack of positivity of the appropriate bilinear functional restricts the analysis within the realm of Schwartz distributions.

In [10,11] we investigated (heterogeneous) velocity averaging for equations of different types in the L^p framework for $p \geqslant 2$. More precisely, we considered a sequence of functions (u_n) weakly converging to zero in the above space, and satisfying the following sequence of (fractional order partial differential) equations:

$$\mathcal{P}u_n(\mathbf{x},\mathbf{p}) = \sum_{k=1}^{d} \partial_{x_k}^{\alpha_k}\big(a_k(\mathbf{x},\mathbf{p})u_n(\mathbf{x},\mathbf{p})\big) = \partial_{\mathbf{p}}^{\boldsymbol{\kappa}} G_n(\mathbf{x},\mathbf{p}), \tag{1}$$

where $\alpha_k > 0$ are real numbers, and $\partial_{x_k}^{\alpha_k}$ are (the Fourier) multiplier operators with the symbols $(2\pi i\xi_k)^{\alpha_k}$, while $\partial_{\mathbf{p}}^{\boldsymbol{\kappa}} = \partial_{p_1}^{\kappa_1}\dots\partial_{p_m}^{\kappa_m}$ for a multi-index $\boldsymbol{\kappa} = (\kappa_1,\dots,\kappa_m) \in \mathbf{N}^m$.

It is well known that the sequence (u_n) does not necessarily converge strongly in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^{m+d})$ for any $p \geqslant 1$. Still, from the viewpoint of applications, it is often enough to analyse the behaviour of the sequence of the solutions averaged with respect to the velocity variable $(\int_{\mathbf{R}^m} \rho(\mathbf{p})u_n(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{p})$, $\rho \in \mathrm{C}_c(\mathbf{R}^m)$ (see, e.g., [6,12]), which can be strongly precompact in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ for an appropriate $p \geqslant 1$, even when the sequence $(u_n(\mathbf{x},\mathbf{p}))$ is not. Such results are usually called velocity averaging lemmas (e.g., [1,7,13,14]).

As we saw in [11, Section 4] (see also sketch of the proof of Theorem 3.1 here), if the coefficients are irregular in the sense that they belong to L^p space for an appropriate $p > 1$, the velocity averaging problem naturally leads to a bilinear functional on $\mathrm{L}^p(\mathbf{R}^d) \otimes \mathrm{C}(\mathrm{P})$, where P is an appropriate manifold (non-necessarily the sphere), and a problem of its extension to $\mathrm{L}^p(\mathbf{R}^d;\mathrm{C}(\mathrm{P}))$.

The main goal of the note is to find conditions under which it is possible to extend a continuous bilinear functional on $\mathrm{L}^p(\mathbf{R}^d) \otimes E$, $p > 1$ and E being a Banach space, to a continuous functional on $\mathrm{L}^p(\mathbf{R}^d;E)$ (Section 2), and to apply it to the velocity averaging theory in order to generalise the results to the case when solutions to (1) belong merely to $L^p(\mathbf{R}^{d+m})$, $p > 1$ (Section 3).

2. Functional analytic tools

In this section we shall introduce analytical tools required to prove the velocity averaging result. We start with the announced theorem on an extension of bilinear functionals on $\mathrm{L}^p(\mathbf{R}^d) \otimes E$.

Theorem 2.1. *Let B be a continuous bilinear functional on $\mathrm{L}^p(\mathbf{R}^d) \otimes E$, where E is a separable Banach space and $p \in \langle 1,\infty\rangle$. Then B can be extended as a continuous functional on $\mathrm{L}^p(\mathbf{R}^d;E)$ if and only if there exists a (non-negative) function $b \in \mathrm{L}^{p'}(\mathbf{R}^d)$ such that for every $\psi \in E$ and almost every $\mathbf{x} \in \mathbf{R}^d$, it holds*

$$\big|\tilde{B}\psi(\mathbf{x})\big| \leqslant b(\mathbf{x})\|\psi\|_E, \tag{2}$$

where $\tilde{B}$ is a bounded linear operator $E \to \mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\langle \tilde{B}\psi,\varphi\rangle = B(\varphi,\psi)$, $\varphi \in \mathrm{L}^p(\mathbf{R}^d)$.

Proof. Let us assume that (2) holds. In order to prove that B can be extended as a linear functional on $\mathrm{L}^p(\mathbf{R}^d;E)$, it is enough to obtain an appropriate bound on the following dense subspace of $\mathrm{L}^p(\mathbf{R}^d;E)$:

$$\left\{ \sum_{i=1}^{N} \psi_i\chi_i(\mathbf{x}):\ \psi_i \in E,\ N \in \mathbf{N} \right\}, \tag{3}$$

where χ_i are characteristic functions associated with mutually disjoint, finite measure sets.

For an arbitrary function $\phi = \sum_{i=1}^N \psi_i\chi_i$ from (3), the bound follows easily once having noticed that

$$\|\phi\|^p_{\mathrm{L}^p(\mathbf{R}^d;E)} = \int_{\mathbf{R}^d} \left\| \sum_{i=1}^{N} \psi_i\chi_i(\mathbf{x}) \right\|_E^p \mathrm{d}\mathbf{x} = \int_{\mathbf{R}^d} \sum_{i=1}^{N} \|\psi_i\|_E^p \chi_i(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

In order to prove the opposite side of the implication, take a countable dense set of functions in the unit sphere of E, and denote them by ψ_j, $j \in \mathbf{N}$. For each function $\tilde{B}\psi_j \in \mathrm{L}^{p'}(\mathbf{R}^d)$ denote by D_j the corresponding set of Lebesgue points, and their intersection by $D = \bigcap_j D_j$.

For any $x \in D$ and $k \in \mathbf{N}$, denote

$$b_k(x) = \max_{1\leqslant j\leqslant k} \tilde{B}\psi_j(x) = \sum_{j=1}^{k} \tilde{B}\psi_j(x)\chi_j^k(x)$$

where $\chi^k_{j_0}$ is characteristic function of the set $X^k_{j_0}$ of all points $x \in D$ for which the above maximum is achieved for $j = j_0$. Furthermore, we can assume that for each k sets X^k_j are mutually disjoint. The sequence (b_k) is clearly monotonic sequence of positive functions, bounded in $\mathrm{L}^{p'}(\mathbf{R}^d)$, whose limit (in the same space) we denote by b. As D is a full measure set we have that for every ψ_j:

$$\left|\tilde{B}\psi_j(\mathbf{x})\right| \leqslant b(\mathbf{x}) \quad \left(\text{a.e. } \mathbf{x} \in \mathbf{R}^d\right).$$

The assertion now follows since (2) holds on the dense set of functions ψ_j, $j \in \mathbf{N}$. □

Next, we shall need multiplier operators with symbols defined on a manifold P determined by the order of the derivatives from (1):

$$\mathrm{P} = \left\{\boldsymbol{\xi} \in \mathbf{R}^d \colon \sum_{k=1}^{d} |\xi_k|^{l\alpha_k} = 1\right\},$$

where l is a minimal number such that $l\alpha_k > d$ for each k. In order to associate an L^p multiplier to a function defined on P, we extend it to $\mathbf{R}^d \backslash \{0\}$ by means of the projection:

$$\left(\pi_{\mathrm{P}}(\boldsymbol{\xi})\right)_i = \xi_i\left(|\xi_1|^{l\alpha_1} + \cdots + |\xi_d|^{l\alpha_d}\right)^{-1/l\alpha_i}, \quad i = 1, \ldots, d, \ \boldsymbol{\xi} \in \mathbf{R}^d \backslash \{0\}.$$

According to the choice of l, given manifolds are at least of class C^d which enables us to introduce an appropriate variant of the H-distributions.

Theorem 2.2. *Let (u_n) be a bounded sequence of functions in $\mathrm{L}^s(\mathbf{R}^{d+m})$, $s \in \langle 1, 2\rangle$, with a common compact support with respect to $\mathbf{p} \in \mathbf{R}^m$ variable, and let (v_n) be a bounded sequence of uniformly compactly supported functions in $\mathrm{L}^\infty(\mathbf{R}^m)$. Then, after passing to a subsequence (not relabelled), for any $\bar{s} \in \langle 1, s\rangle$ there exists a continuous bilinear functional B on $\mathrm{L}^{\bar{s}'}(\mathbf{R}^{d+m}) \otimes \mathrm{C}^d(\mathrm{P})$ such that for every $\varphi \in \mathrm{L}^{\bar{s}'}(\mathbf{R}^{d+m})$ and $\psi \in \mathrm{C}^d(\mathrm{P})$ it holds*

$$B(\varphi, \psi) = \lim_{n\to\infty} \int_{\mathbf{R}^{d+m}} \varphi(\mathbf{x}, \mathbf{p}) u_n(\mathbf{x}, \mathbf{p}) (\mathcal{A}_{\psi_{\mathrm{P}}} v_n)(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{p}, \tag{4}$$

where $\mathcal{A}_{\psi_{\mathrm{P}}}$ is the (Fourier) multiplier operator on $\mathbf{R}^d$ associated with $\psi \circ \pi_{\mathrm{P}}$.

Proof. According to the Marcinkiewicz multiplier theorem [9, Theorem 5.2.4] and the Hölder inequality, we conclude that the right-hand side of (4) determines a sequence of bilinear mappings (B_n) uniformly bounded by $C\|\psi\|_{C^d(\mathrm{P})}\|\varphi\|_{L^{\bar{s}'}(\mathbf{R}^{d+m})}$ for a constant C independent of ψ or φ. The statement now follows from [3, Lemma 3.2]. □

According to the Schwartz kernel theorem, the functional B defined above can be extended as a distribution from $\mathcal{D}'(\mathbf{R}^{d+m} \times \mathrm{P})$. However, by means of Theorem 2.1 we get a better result.

Corollary 2.3. *The bilinear functional B defined in Theorem 2.2 can be extended as a continuous functional on $\mathrm{L}^{\bar{s}'}(\mathbf{R}^{d+m}; \mathrm{C}^d(\mathrm{P}))$.*

3. Application to the velocity averaging

In this section, we consider a sequence of solutions u_n to (1), weakly converging to zero in $\mathrm{L}^s(\mathbf{R}^{d+m})$ for some $s > 1$. Without loss of generality, we assume that (u_n) is uniformly compactly supported with respect to $\mathbf{p} \in \mathbf{R}^m$. Furthermore, let us assume that coefficients entering the equation satisfy the following conditions:

(a) $a_k \in \mathrm{L}^{\bar{s}'}(\mathbf{R}^{d+m})$, for some $\bar{s} \in \langle 1, s\rangle$, $k = 1, \ldots, d$,
(b) the sequence (G_n) is strongly precompact in the anisotropic space $\mathrm{L}^{\hat{s}'}(\mathbf{R}^m; \mathrm{W}^{-\boldsymbol{\alpha}, \hat{s}'}(\mathbf{R}^d))$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and $1/\bar{s}' + 1/\hat{s} = 1/s'$.

The following, velocity averaging result holds.

Theorem 3.1. *Let $A = \sum_k (2\pi i\xi_k)^{\alpha_k} a_k(\mathbf{x}, \mathbf{p})$ be the principal symbol of the (pseudo-)differential operator $\mathcal{P}$ in (1). Assume that*

$$\frac{|A|^2}{|A|^2 + \delta} \to 1 \quad \text{in } \mathrm{L}^{\bar{s}'}_{\mathrm{loc}}\left(\mathbf{R}^{d+m}; \mathrm{C}^d(\mathrm{P})\right) \tag{5}$$

strongly as $\delta \to 0$. Then, for any $\rho \in \mathrm{C}_c(\mathbf{R}^m)$, the sequence $\int_{\mathbf{R}^m} \rho(\mathbf{p}) u_n(\cdot, \mathbf{p}) \, \mathrm{d}\mathbf{p}$ strongly converges to zero in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$.

Proof. Fix $\rho \in C_c^1(\mathbf{R}^m)$ and $\chi \in L_c^\infty(\mathbf{R}^d)$, and denote by V a weak $*$ $\mathrm{L}^\infty(\mathbf{R}^d)$ limit along some subsequence (not relabelled) of the sequence of functions:

$$V_n = \frac{\chi(\mathbf{x})\overline{\int_{\mathbf{R}^m}\rho(\mathbf{q})u_n(\mathbf{x},\mathbf{q})\,\mathrm{d}\mathbf{q}}}{|\int_{\mathbf{R}^m}\rho(\mathbf{q})u_n(\mathbf{x},\mathbf{q})\,\mathrm{d}\mathbf{q}|}.$$

Denote $v_n = V_n - V$ and remark that $v_n \overset{*}{\rightharpoonup} 0$ in $\mathrm{L}^\infty(\mathbf{R}^d)$.

The proof is accomplished by showing that the H-distribution B from Theorem 2.2 associated with the sequences (u_n) and (v_n) equals zero. By repeating the procedure from the beginning of [11, Section 4], we conclude that it holds

$$\langle fA, B\rangle = 0, \quad f \in \mathrm{C_c}\big(\mathbf{R}^{d+m}\big)\otimes \mathrm{C}^d(\mathrm{P}). \tag{6}$$

According to Corollary 2.3, the distribution B can be tested on functions from the space $\mathrm{L}^{\bar{s}'}(\mathbf{R}^{d+m};\mathrm{C}^d(\mathrm{P}))$. Thus, for an arbitrary $\phi \in \mathcal{D}(\mathbf{R}^{d+m}\times\mathrm{P})$, we can choose in (6) a test function of the form:

$$f(\mathbf{x},\mathbf{p},\boldsymbol{\xi}) = \frac{\phi(\mathbf{x},\mathbf{p},\boldsymbol{\xi})\overline{A(\mathbf{x},\mathbf{p},\boldsymbol{\xi})}}{|A(\mathbf{x},\mathbf{p},\boldsymbol{\xi})|^2+\delta},$$

for any fixed $\delta > 0$. By passing to the limit in such obtained (6) and using (5), we conclude

$$B = 0.$$

In order to finish the proof, take in (4) test functions $\psi = 1$ and $\varphi(\mathbf{x},\mathbf{p}) = \chi(\mathbf{x})\rho(\mathbf{p})$ for the previously chosen ρ and χ. Since $B = 0$, from the definition of the sequence (v_n) (keep also in mind that $u_n \rightharpoonup 0$ in $\mathrm{L}^s(\mathbf{R}^{d+m})$), it follows

$$\lim_{n\to\infty}\int_{\mathbf{R}^d}\chi^2(\mathbf{x})\left|\left(\int_{\mathbf{R}^m}\rho(\mathbf{p})u_n(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{p}\right)\right|\mathrm{d}\mathbf{x} = 0,$$

which concludes the proof (due to arbitrariness of ρ and χ). □

A special case of conditions (5) are the following non-degeneracy conditions:

For $U^\delta = \{(\mathbf{x},\mathbf{p}):\ A^2(\mathbf{x},\boldsymbol{\xi},\mathbf{p}) > \delta,\ \boldsymbol{\xi}\in\mathrm{P}\}$ and every compact set $K \subset \mathbf{R}^{d+m}$, the measure of $K\backslash U^\delta$ goes to 0 when $\delta \to 0$.

It is not difficult to see that given non-degeneracy conditions are satisfied for elliptic and parabolic equations, but also fractional convection–diffusion equations [5], and parabolic equations with a fractional time derivative [4] that degenerate on a set of measure zero.

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