C. R. Acad. Sci. Paris, Ser. I 350 (2012) 1007–1009

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Lie Algebras

L_0-types common to a Borel–de Siebenthal discrete series and its associated holomorphic discrete series

L_0-types communs à une série discrète de Borel–de Siebenthal et sa série discrète holomorphe associée

Pampa Paul, K.N. Raghavan, Parameswaran Sankaran

The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India

ARTICLE INFO

Article history:
Received 1 October 2012
Accepted after revision 13 November 2012
Available online 21 November 2012

Presented by the Editorial Board

ABSTRACT

Let G_0 be a simply connected non-compact real simple Lie group and let K_0 be a maximal compact subgroup of G_0. Suppose that K_0 is semisimple and that $\mathrm{rank}(K_0) = \mathrm{rank}(G_0)$. Let Δ^+ be a Borel–de Siebenthal positive root system and let π_λ be the Borel–de Siebenthal discrete series of G_0 with Harish-Chandra parameter λ. One has a certain subgroup $L_0 \subset K_0$ so that K_0/L_0 is an irreducible Hermitian symmetric space. Also, there is a holomorphic discrete series $\pi_{\lambda'}$ of K_0^*, the non-compact dual of K_0, with Harish-Chandra parameter $\lambda' := \lambda - (1/2)\sum\alpha$, where the sum is over non-compact roots in Δ^+. We prove that there are infinitely many L_0-types common to π_λ and $\pi_{\lambda'}$ under certain hypotheses.

RÉSUMÉ

Soit G_0 un groupe de Lie simple réel simplement connexe non-compact et soit K_0 un sous-groupe compact maximal de G_0. Supposons que K_0 soit semisimple, et que $\mathrm{rang}(K_0) = \mathrm{rang}(G_0)$. Supposons que Δ^+ soit un système positif de racines de Borel–de Siebenthal de G_0. Soit π_λ la représentation de la série discrète de Borel–de Siebenthal de G_0 avec paramètre de Harish-Chandra λ. Il existe un sous-groupe connexe $L_0 \subset K_0$ tel que K_0/L_0 soit un espace Hermitien symétrique irréductible. Soit K_0^* le dual non-compact de K_0 par rapport à L_0. On a une série discrète holomorphe $\pi_{\lambda'}$ de K_0^* avec paramètre de Harish-Chandra $\lambda' := \lambda - (1/2)\sum\alpha$ où α parcourt les racines non-compactes de Δ^+. On montre qu'il existe une infinité de L_0-types communs à π_λ et $\pi_{\lambda'}$ sous certaines hypothèses.

1. Introduction

Let G_0 be a simply connected non-compact real simple Lie group and K_0 be a maximal compact subgroup of G_0. Let $T_0 \subset K_0$ be a maximal torus. Assume that $\mathrm{rank}(K_0) = \mathrm{rank}(G_0)$ so that G_0 has discrete series representations. Note that T_0 is a Cartan subgroup of G_0 as well. We shall denote by $\mathfrak{g}_0$, $\mathfrak{k}_0$, and $\mathfrak{t}_0$ the Lie algebras of G_0, K_0, and T_0 respectively and by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{t}$ the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, and $\mathfrak{t}_0$ respectively. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$. Let Δ^+ be a Borel–de Siebenthal positive root system so that the set of simple roots Ψ has exactly one non-compact root ν. When G_0/K_0

E-mail addresses: pampa@imsc.res.in (P. Paul), knr@imsc.res.in (K.N. Raghavan), sankaran@imsc.res.in (P. Sankaran).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.11.009

is a Hermitian symmetric space, one has the holomorphic discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$ where γ is an integral weight which is non-negative on compact simple roots and $\gamma+\rho_{\mathfrak{g}}$ is negative on non-compact positive roots.

Assume that G_0/K_0 is *not* a Hermitian symmetric space. This is equivalent to the requirement that the centre of K_0 is discrete. Let $L_0 \subset K_0$ be the reductive subgroup containing T_0 and having root system $\Delta_0 \subset \Delta$ generated by the set of compact simple roots $\Psi \setminus \{\nu\}$. The homogeneous space K_0/L_0 is an irreducible compact Hermitian symmetric space. It turns out that G_0/L_0 is a *flag domain*: It is an open complex manifold embeddable in a flag variety G/Q where G is a simply connected Lie group with $Lie(G)=\mathfrak{g}$ and $Q \subset G$ the parabolic subgroup corresponding to the parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{n}_-$, where $\mathfrak{l}=\mathfrak{t}+\sum_{\alpha\in\Delta_0}\mathfrak{g}_\alpha$ and $\mathfrak{n}_-=\sum_{\alpha\in\Delta^+\setminus\Delta_0}\mathfrak{g}_{-\alpha}$, so that the imbedding $K_0/L_0 \subset G_0/L_0$ is holomorphic.

Recall that G_0/K_0 is assumed to be a non-Hermitian symmetric space. The Borel–de Siebenthal discrete series of G_0, whose study was initiated by Ørsted and Wolf [3], is defined analogously to the holomorphic discrete series as follows: Let γ be the highest weight of a finite-dimensional irreducible representation of L_0 such that $\gamma+\rho_{\mathfrak{g}}$ is negative on all positive roots of $\mathfrak{g}$ complementary to those of $\mathfrak{l}$. Here $\rho_{\mathfrak{g}}$ denotes half the sum of positive roots of $\mathfrak{g}$. The Borel–de Siebenthal discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is the discrete series representation of G_0 for which the Harish-Chandra parameter is $\gamma+\rho_{\mathfrak{g}}$. It is assumed that γ is analytically integrable with respect to $p(L_0)$ where $p: G_0 \to G$ is the homomorphism that induces $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$. Ørsted and Wolf proved that the K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is in fact K_1-admissible, where K_1 is a certain simple factor of K_0, and described the K_0-finite part in terms of the Dolbeault cohomology as $\bigoplus_{m\geqslant 0} H^s(K_0/L_0; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))$ where $s=\dim_{\mathbb{C}} K_0/L_0$, $\mathbb{E}_\gamma$ denotes the holomorphic vector bundle associated to the irreducible (finite-dimensional) L_0-module E_γ with highest weight γ, $\mathfrak{u}_{-1}$ is a certain irreducible finite-dimensional complex representation of L_0 such that the associated holomorphic vector bundle over K_0/L_0 is the conormal bundle to the imbedding of $K_0/L_0 \subset G_0/L_0$, and $\mathbb{S}^m(\mathfrak{u}_{-1})$ denotes the vector bundle associated to the m-th symmetric power $S^m(\mathfrak{u}_{-1})$ of $\mathfrak{u}_{-1}$. R. Parthasarathy [4] obtained the same description in a more general context using entirely different techniques.

Let $\Delta_i \subset \Delta$, $-2 \leqslant i \leqslant 2$ denote that set of all roots such that, when expressed as a sum of simple roots, the coefficient of ν equals i. Let $\Delta_0^{\pm}=\Delta^{\pm}\cap\Delta_0$. Then $\Delta^+=\Delta_0^+\cup\Delta_1\cup\Delta_2$. The root system of $\mathfrak{k}$ is $\Delta_{\mathfrak{k}}=\Delta_0\cup\Delta_2^{\pm}$, and the induced positive system of $(\mathfrak{k},\mathfrak{t})$ is obtained as $\Delta_{\mathfrak{k}}^+=\Delta_0^+\cup\Delta_2$.

Let γ be the highest weight of a $p(L_0)$-representation such that $\gamma+\rho_{\mathfrak{g}}$ is positive on Δ_0^+ and negative on $\Delta_1\cup\Delta_2$ so that we have the Borel–de Siebenthal discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$. Let $(K_0^*, p(L_0))$ denote the Hermitian symmetric pair dual to the pair (K_0, L_0). The set of non-compact roots in $\Delta_{\mathfrak{k}}^+$ equals Δ_2 with respect to the real form $Lie(K_0^*)$ of $\mathfrak{k}$. We also have a holomorphic discrete series of K_0^*, with Harish-Chandra parameter $\gamma+\rho_{\mathfrak{k}}$, denoted $\pi_{\gamma+\rho_{\mathfrak{k}}}$. See Section 2. It is a natural question to ask which L_0-types are common to the Borel–de Siebenthal discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$ and the corresponding holomorphic discrete series $\pi_{\gamma+\rho_{\mathfrak{k}}}$. We shall answer this question completely when $\mathfrak{k}_1=\mathfrak{su}(2)$, the so-called quaternionic case. See Theorem 1.1. In the non-quaternionic case, we obtain complete results assuming that (i) there exists a non-trivial one-dimensional L_0-subrepresentation in the symmetric algebra $S^*(\mathfrak{u}_{-1})$ and (ii) the longest element of the Weyl group of K_0 preserves Δ_0. See Theorem 1.2 below. Note that the second condition is trivially satisfied in the quaternionic case. The existence of non-trivial one-dimensional L_0-submodule in the symmetric algebra $S^*(\mathfrak{u}_{-1})$ greatly simplifies the task of detecting occurrence of common L_0-types. The classification of Borel–de Siebenthal positive systems for which such one-dimensional submodules exist has been carried out by Ørsted and Wolf [3, §4].

We now state the main results of this Note:

Theorem 1.1. *We keep the above notations. Suppose that Lie*$(K_1)\cong\mathfrak{su}_2$. *If* $\mathfrak{g}_0=\mathfrak{so}(4,1)$ *or* $\mathfrak{sp}(1,l-1)$, $l>1$, *then there are at most finitely many* L_0*-types common to* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *and* $\pi_{\gamma+\rho_{\mathfrak{k}}}$. *Moreover, if* $\dim E_\gamma=1$, *then there are no common* L_0*-types.*

Suppose that $\mathfrak{g}_0\neq\mathfrak{so}(4,1)$ *or* $\mathfrak{sp}(1,l-1)$, $l>1$. *Then each* L_0*-type in the holomorphic discrete series* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *occurs in the Borel–de Siebenthal discrete series* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with* infinite *multiplicity.*

The case $G_0=SO(4,1)$, $Sp(1,l-1)$ are exceptional among the quaternionic cases in that these are precisely the cases for which the prehomogeneous space $(L,\mathfrak{u}_1)$ have no (non-constant) relative invariants—equivalently $S^m(\mathfrak{u}_{-1})$, $m\geqslant 1$, have no one-dimensional L_0-subrepresentations. In the non-quaternionic case, we have the following result:

Theorem 1.2. *With the above notations, suppose that* $w_{\mathfrak{k}}^0$, *the longest element of the Weyl group of* $(\mathfrak{k},\mathfrak{t})$ *(with respect to* $\Delta_{\mathfrak{k}}^+$*), preserves* Δ_0 *and that there exists a one-dimensional* L_0*-submodule in* $S^m(\mathfrak{u}_{-1})$ *for some* $m\geqslant 1$. *Then there is an infinite family of* L_0*-types common to* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *and* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *each of which occurs in* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with infinite multiplicity. Moreover, if* $\dim E_\gamma=1$, *then* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *itself occurs in* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with infinite multiplicity.*

The existence (or non-existence) of one-dimensional L_0-submodules in $\bigoplus_{m\geqslant 1} S^m(\mathfrak{u}_{-1})$ is closely related to the L_0-admissibility of $\pi_{\gamma+\rho_{\mathfrak{g}}}$. Note that Theorem 1.1 implies that, when $\mathfrak{k}_1=\mathfrak{su}(2)$, the restriction of the Borel–de Siebenthal discrete series is not L_0-admissible when $\mathfrak{g}_0\neq\mathfrak{so}(4,1)$, $\mathfrak{sp}(1,l-1)$. It turns out that other than these two exceptional cases, in each of the remaining (quaternionic) cases, there exists a one-dimensional subrepresentation in $\bigoplus_{m>0} S^m(\mathfrak{u}_{-1})$. When $\mathfrak{g}_0=\mathfrak{so}(4,1)$, $\mathfrak{sp}(1,l-1)$, $l>1$, the Borel–de Siebenthal discrete series is L_0-admissible. In fact we shall establish the following result:

Proposition 1.3. *Suppose that $S^m(\mathfrak{u}_{-1})$ has a one-dimensional L_0-subrepresentation for some $m \geqslant 1$, then the Borel–de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ is not $[L_0, L_0]$-admissible. The converse holds if $\mathfrak{k}_1 = \mathfrak{su}(2)$.*

Combining Theorems 1.1 and 1.2, we see that there are infinitely many L_0-types common to $\pi_{\gamma+\rho_\mathfrak{g}}$ and $\pi_{\gamma+\rho_\mathfrak{k}}$ whenever $S^m(\mathfrak{u}_{-1})$ has a one-dimensional L_0-submodule for some $m \geqslant 1$ and $w^0_\mathfrak{k}(\Delta_0) = \Delta_0$. We are led to the following questions.

Questions. Suppose that there exist infinitely many common L_0-types between a Borel–de Siebenthal discrete series representation $\pi_{\gamma+\rho_\mathfrak{g}}$ of G_0 and the holomorphic discrete series representation $\pi_{\gamma+\rho_\mathfrak{k}}$ of K_0^*. (i) Does there exist a one-dimensional L_0-subrepresentation in $S^m(\mathfrak{u}_{-1})$? (ii) Is it true that $w^0_\mathfrak{k}(\Delta_0) = \Delta_0$?

We make use of the description of the K_0-finite part of the Borel–de Siebenthal discrete series obtained by Ørsted and Wolf, in terms of the Dolbeault cohomology of the flag variety K_0/L_0 with coefficients in the holomorphic bundle associated to the L_0-representation $E_\gamma \otimes S^m(\mathfrak{u}_{-1})$. Proof of Theorem 1.1 involves only elementary considerations. Proof of Theorem 1.2 crucially makes use of a result of Schmid [6] on the decomposition of the L_0-representation $S^m(\mathfrak{u}_{-2})$. Another ingredient of the proof is Littelmann's Branching Rule [2] describing the restriction of a K_0-representation to L_0.

There are three major obstacles in obtaining complete result in the non-quaternionic case. The first is the decomposition of $S^m(\mathfrak{u}_{-1})$ into L_0-types E_λ. Secondly, one has the problem of decomposing of the tensor product $E_\gamma \otimes E_\lambda$ into irreducible L_0-representations E_κ. Finally, one has the restriction problem of decomposing the irreducible K_0-representation $H^s(K_0/L_0; \mathbb{E}_\kappa)$ into L_0-subrepresentations. The latter two problems can, in principle, be solved using the work of Littelmann [2]. The problem of detecting occurrence of an infinite family of common L_0-types in the general case appears to be difficult.

The detailed proofs of the above results will be published elsewhere [5].

2. Holomorphic discrete series associated to a Borel–de Siebenthal discrete series

We keep the notations of Section 1. Recall that K_0/L_0 is an irreducible compact Hermitian symmetric space. Let K_0^* be the dual of K_0 in K with respect to $p(L_0)$ so that $K_0^*/p(L_0)$ is the non-compact irreducible Hermitian symmetric space dual to K_0/L_0. Note that $\mathfrak{k} = Lie(K_0^*) \otimes_\mathbb{R} \mathbb{C}$ and that $\mathfrak{t} \subset \mathfrak{l}$ is a Cartan subalgebra of $\mathfrak{k}$. The sets of compact and non-compact roots of $(\mathfrak{k}, \mathfrak{t})$ are Δ_0 and $\Delta_2 \cup \Delta_{-2}$ respectively. The positive system $\Delta^+_\mathfrak{k}$ is a Borel–de Siebenthal positive system of K_0^*.

Let $\gamma + \rho_\mathfrak{g}$ be the Harish-Chandra parameter for a Borel–de Siebenthal discrete series of G_0. Thus γ is the highest weight of an irreducible representation of $p(L_0)$ and $\langle \gamma + \rho_\mathfrak{g}, \beta \rangle < 0$ for all $\beta \in \Delta_1 \cup \Delta_2$.

Clearly $\langle \gamma + \rho_\mathfrak{k}, \alpha \rangle > 0$ for all positive compact roots $\alpha \in \Delta^+_0$. We claim that $\langle \gamma + \rho_\mathfrak{k}, \beta \rangle < 0$ for all positive non-compact roots $\beta \in \Delta_2$. To see this, let $\beta_i \in \Delta_i$, $i = 1, 2$. Observe that $\beta_1 + \beta_2$ is not a root and so $\langle \beta_1, \beta_2 \rangle \geqslant 0$. It follows that $\langle \rho_\mathfrak{k}, \beta_2 \rangle = \langle \rho_\mathfrak{g} - 1/2 \sum_{\beta_1 \in \Delta_1} \beta_1, \beta_2 \rangle = \langle \rho_\mathfrak{g}, \beta_2 \rangle - 1/2 \sum_{\beta_1 \in \Delta_1} \langle \beta_1, \beta_2 \rangle \leqslant \langle \rho_\mathfrak{g}, \beta_2 \rangle$. So $\langle \gamma + \rho_\mathfrak{k}, \beta \rangle \leqslant \langle \gamma + \rho_\mathfrak{g}, \beta \rangle < 0$ for all $\beta \in \Delta_2$. Thus, by [1, Theorem 6.6, Chapter VI], $\gamma + \rho_\mathfrak{k}$ is the Harish-Chandra parameter for a holomorphic discrete series $\pi_{\gamma+\rho_\mathfrak{k}}$ of K_0^*, which is naturally associated to the Borel–de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ of G_0.

References

[1] A.W. Knapp, Representation Theory of Semisimple Groups. An Overview Based on Examples, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001, reprint of the 1986 original.

[2] P. Littelmann, A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras, Invent. Math. 116 (1994) 329–346.

[3] B. Ørsted, J.A. Wolf, Geometry of the Borel–de Siebenthal discrete series, J. Lie Theory 20 (1) (2010) 175–212.

[4] R. Parthasarathy, An algebraic construction of a class of representations of a semi-simple Lie algebra, Math. Ann. 226 (1) (1977) 1–52.

[5] P. Paul, K.N. Raghavan, P. Sankaran, L_0-types common to a Borel–de Siebenthal discrete series and its associated holomorphic discrete series, arXiv:1210.0123.

[6] W. Schmid, Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, Invent. Math. 9 (1969/1970) 61–80.