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# On stable solutions of the finite non-periodic Toda lattice

## *Sur les solutions stables du réseau de Toda non périodique fini*

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ABSTRACT

In this Note we study stable solutions of the finite non-periodic ($A_n$-type) Toda lattice. Solutions of the $A_n$-type Toda lattice are obtained by Gauss decomposition. Such solutions are unstable because the Gauss decomposition brings singularities. We obtain stable solutions which are entire functions on $\mathbb{R}$ as the soliton solutions by modified Gauss decomposition.

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RÉSUMÉ

On étudie les solutions stables du réseau de Toda fini non périodique de type $A_n$. On obtient des solutions du réseau de Toda de type $A_n$ par décomposition de Gauss. Les solutions ainsi obtenues sont instables car la décomposition de Gauss possède des singularités. Les solutions stables obtenues par la méthode de décomposition de Gauss modifiée sont des fonctions entières sur $\mathbb{R}$, elles sont des solutions-solitons.

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### Version française abrégée

On sait que le réseau de Toda infini $\ddot{q} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$, $n \in \mathbb{Z}$ admet des solutions-solitons, par exemple $\tau_n(t) = 1 + e^{2(kn-\omega t)}$ (solution 1-soliton), $k$ et $\omega$ vérifient la relation de dispersion $\omega^2 = \sinh^2 k$. Comme tout champ du réseau de Toda s'écrit sous la forme d'une somme finie de termes $\frac{\partial}{\partial t} \log \tau_n(t)$, $n \in \mathbb{Z}$ ; on voit que les singularités des solutions-solitons ne sont pas sur $\mathbb{R}$. Les solutions-solitons du réseau de Toda infini ont été construites Par Hirota par une méthode directe [4]. Les solutions sont également obtenues par la méthode de dispersion inverse [1,2]. Dans le cas du réseau de Toda fini non périodique (réseau de Toda de type $A_n$), les solutions sont obtenues par décomposition de Gauss $W_\infty(t)^{-1} W_0(t) = e^{tL(0)} A$ où $W_\infty(t) \in \bar{N}$ (le sous-groupe unipotent triangulaire inférieur), $W_0(t) \in B$ (le sous-groupe unipotent triangulaire supérieur), et $A \in GL_n(\mathbb{R})$ est une matrice constante. Les solutions obtenues par la décomposition de Gauss précédente possèdent une singularité en $t = t_0$ où la décomposition de Gauss devient impossible. L'analyse de Painlevé du réseau de Toda a été faite par Flaschka et Haine [3]. Dans [5] on a donné un exemple de solution méromorphe de réseau de Toda de type $A_n$ ayant une singularité sur $\mathbb{R}$. L'objet de cette Note est la construction de solutions du réseau de Toda de type $A_n$ n'ayant aucune singularité sur $t \in \mathbb{R}$ comme solutions-solitons. On appelle ces solutions, solutions stables. A titre d'exemple, de solution stable, on donne la solution 1-soliton. On pense que ces solutions stables pourront être utiles dans l'étude du circuit $LC$ non linéaire.

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## 1. Preliminaries

It is known that the infinite Toda lattice $\ddot{q}_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}}$, $n \in \mathbb{Z}$ has the soliton solutions for example $\tau_n(t) = 1 + e^{2(kn-\omega t)}$ (1-soliton solution). $k$ and $\omega$ must satisfy the dispersion relation $\omega^2 = \sinh^2 k$. Since all the fields of the Toda lattice are written by finite sum of $-\mathrm{d}/\mathrm{d}t \log \tau_n(t)$, $n \in \mathbb{Z}$, we see that there are no singularities on $t \in \mathbb{R}$ for soliton solutions of the infinite Toda lattice. In the case of the finite non-periodic Toda lattice ($A_n$-type Toda lattice), the solutions are obtained by the Gauss decomposition $W_\infty(t)^{-1} W_0(t) = e^{tL(0)} A$, where $W_\infty(t) \in \bar{N}$ (the subgroup of lower triangular unipotent matrices) and $W_0(t) \in B$ (upper triangular Borel subgroup) and $A \in GL_n(\mathbb{R})$ is a constant matrix. The solutions obtained by above Gauss decomposition have a singularity at $t = t_0$ where $e^{tL(0)} A$ cannot be decomposed. The purpose of this Note is to construct solutions of $A_n$-type Toda lattice which have no singularities on $t \in \mathbb{R}$ as soliton solutions. We call such solutions, stable solutions. We give 1-soliton solution as an example of stable solution.

We consider the $A_n$-type non-periodic Toda lattice (open boundary problem)

$$\begin{cases} \ddot{q}_1 = -e^{q_1-q_2}, \\ \ddot{q}_j = e^{q_{j-1}-q_j} - e^{q_j-q_{j+1}}, \quad j = 2, \ldots, n-1, \\ \ddot{q}_n = e^{q_{n-1}-q_n}. \end{cases} \tag{1}$$

Eq. (1) has the Lax form

$$\dot{L}(t) = \big[\big(L(t)\big)_+, L(t)\big], \tag{2}$$

where $L(t)$ is a tri-diagonal matrix

$$\begin{pmatrix} L_{1,1}(t) & 1 & 0 & \ldots \\ L_{2,1}(t) & L_{2,2}(t) & 1 & \ldots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & L_{n,n-1}(t) & L_{n,n}(t) \end{pmatrix}$$

and $(\ )_+$ is the projection from $\mathfrak{gl}_n(\mathbb{R})$ to $Lie B$. The Lax operators of symmetric type are dealt with [1,2] and non-symmetric type such as our case are dealt with [6]. Put $M = L(0)$. The following Gauss decomposition gives the flow of the Toda lattice (2) and, therefore, (1)

$$W_\infty(t)^{-1} W_0(t) = e^{tM} A, \tag{3}$$

where $W_\infty(t) \in \bar{N}$, $W_0(t) \in B$ and $A \in GL_n(\mathbb{R})$ is a constant matrix. Put $\Phi(t) = e^{tM} A (= (\Phi_{i,j}(t)))$. The $\tau$-functions of the Toda lattice are defined by $\tau_j(t) = |\Phi_{k,\ell}(t)|_{1 \leqslant k,\ell \leqslant j-1}$ determinants of the matrices $(\Phi_{k,\ell}(t))_{1 \leqslant k,\ell \leqslant j-1}$, $j = 2, \ldots, n$. Denote $W_\infty(t) = (w^\infty_{i,j}(t))_{1 \leqslant i,j \leqslant n}$. From the Gauss decomposition (3), we have

$$w^\infty_{j,j-1}(t) = -\mathrm{d}/\mathrm{d}t \log \tau_j(t) + m_{1,1} + \cdots + m_{j-1,j-1}, \quad j = 2, \ldots, n, \tag{4}$$

where $M = (m_{i,j})_{1 \leqslant i,j \leqslant n}$. Furthermore we have

$$\begin{cases} L_{1,1}(t) = \mathrm{d}/\mathrm{d}t \log \tau_2(t) - m_{1,1}, \\ L_{j,j}(t) = \mathrm{d}/\mathrm{d}t \log\big(\tau_{j+1}(t)/\tau_j(t)\big) - m_{j,j}, \qquad 2 \leqslant j \leqslant n-1, \\ L_{n,n}(t) = \mathrm{d}/\mathrm{d}t \log\big(1/\tau_n(t)\big) + m_{1,1} + \cdots + m_{n-1,n-1}. \end{cases} \tag{5}$$

## 2. Stable solutions of the Toda lattice

We consider the modified Gauss decomposition for constant matrices $A \in GL_n(\mathbb{R})$ and $A' \in Mat_n(\mathbb{R})$

$$W_\infty(t)^{-1} W_0(t) = e^{tM} A + A', \tag{6}$$

where $W_\infty(t) \in \bar{N}$ and $W_0(t) \in B$. We consider the orbit $\mathcal{O}_{MT}$ defined by

$$\mathcal{O}_{MT} := \big\{ W_\infty(t) \bigm| W_\infty(t) \text{ satisfies (6)} \big\} \subset \bar{N}.$$

Let $(\ )_-$ be the projection from $\mathfrak{gl}_n(\mathbb{R})$ to $Lie\bar{N}$. The subsector $\mathcal{O}_T$ of $\mathcal{O}_{MT}$ is defined by

$$\mathcal{O}_T := \big\{ W_\infty(t) \bigm| W_\infty(t) \text{ satisfies (6) and } \big(W_\infty(t) M A' W_0(t)^{-1}\big)_- = 0 \big\} \subset \bar{N}.$$

**Lemma.** *The flow on $\mathcal{O}_{MT}$ reduces to the flow of the ordinary Toda lattice on $\mathcal{O}_T$.*

**Proof.** From (6), we obtain

$$-\dot{W}_\infty(t)W_\infty(t)^{-1} + \dot{W}_0(t)W_0(t)^{-1} = W_\infty(t)Me^{tM}AW_0(t)^{-1}. \tag{7}$$

Since $e^{tM}AW_0(t)^{-1} = W_\infty(t)^{-1} - A'W_0(t)^{-1}$, we have

$$-\dot{W}_\infty(t)W_\infty(t)^{-1} + \dot{W}_0(t)W_0(t)^{-1} = W_\infty(t)MW_\infty(t)^{-1} - W_\infty(t)MA'W_0(t)^{-1}. \tag{8}$$

Suppose $(W_\infty(t)MA'W_0(t)^{-1})_- = 0$, we have

$$\dot{W}_\infty(t) = -\big(W_\infty(t)MW_\infty(t)^{-1}\big)_- W_\infty(t). \tag{9}$$

Thus $L(t) = W_\infty(t)MW_\infty(t)^{-1}$ satisfies (2). $\square$

We assume that $M$ has $n$ distinct nonzero real eigenvalues $\alpha_1, \dots, \alpha_n$ and $M$ is diagonalized by $\Xi = (\xi_{i,j})_{1\leqslant i,j\leqslant n} = (\vec{\xi}_1, \dots, \vec{\xi}_n)$. Put $A = \Xi$ and $A' = \Xi' = (\xi'_{i,j})_{1\leqslant i,j\leqslant n} = (\vec{\xi}'_1, \dots, \vec{\xi}'_n)$ in (6). Then the modified Gauss decomposition (6) becomes

$$W_\infty(t)^{-1}W_0(t) = \Xi \begin{pmatrix} e^{t\alpha_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & e^{t\alpha_n} \end{pmatrix} + \Xi' = \big(e^{t\alpha_1}\vec{\xi}_1 + \vec{\xi}'_1, \dots, e^{t\alpha_n}\vec{\xi}_n + \vec{\xi}'_n\big). \tag{10}$$

Note that if $M$ and, therefore, $\Xi$ are given, then the subsector $\mathcal{O}_T$ is determined by $\Xi'$ from the equation $(W_\infty(t)M\Xi'W_0(t)^{-1})_- = 0$.

**Theorem.** *Suppose that $W_\infty(t)$ and $W_0(t)$ satisfy* (10)*. There exists $\Xi'$ such that if $(W_\infty(0)^{-1}M\Xi'W_0(0)^{-1})_- = 0$ then $(W_\infty(t)M\Xi'W_0(t)^{-1})_- = 0$ for all $t \in \mathbb{R}$.*

**Proof.** At first we show if $(W_\infty(0)^{-1}M\Xi'W_0(0))_- = 0$, there exists $\epsilon > 0$ such that $W_\infty(t)$ satisfies (9) for $|t| < \epsilon$. Put $t = 0$ at (8) where $A = \Xi$ and $A' = \Xi'$. We have

$$-\dot{W}_\infty(0)W_\infty(0)^{-1} + \dot{W}_0(0)W_0(0)^{-1} = W_\infty(0)MW_\infty(0)^{-1} - W_\infty(0)M\Xi'W_0(0)^{-1}.$$

Taking $(\ )_-$ of both sides of above, we have

$$\dot{W}_\infty(0) = -\big(W_\infty(0)MW_\infty(0)^{-1}\big)_- W_\infty(0). \tag{11}$$

Put $W_\infty(t) = W_\infty(0) + tU_1 + t^2U_2 + \cdots$. Suppose that $W_\infty(t)$ satisfies (9). Then we have

$$U_1 = -\big(W_\infty(0)MW_\infty(0)^{-1}\big)_- W_\infty(0), \tag{12}$$

$$2U_2 = \big(W_\infty(0)MW_\infty(0)^{-1}U_1W_\infty(0)^{-1} - U_1M\big)_- W_\infty(0) - \big(W_\infty(0)MW_\infty(0)^{-1}\big)_- U_1, \tag{13}$$

$$\dots$$

Since $U_1 = \dot{W}_\infty(0)$, then (12) is satisfied from (11). Furthermore $U_2, U_3, \dots$ are obtained recursively. In the case of $\Xi' = 0$ which gives ordinary Gauss decomposition, we can obtain analytic solution of (9). We see that $W_\infty(0)$ is determined by $\Xi'$ analytically from the condition

$$\big(W_\infty(0)M\Xi'W_0(0)^{-1}\big)_- = 0.$$

This implies the existence of $\epsilon > 0$ such that $\epsilon = 1/\limsup^n \sqrt{|U_n|} > 0$, where $|U_n|$ is a certain norm of the matrix and we obtain analytic solution of (9) for $|t| < \epsilon$. Eq. (9) implies $(W_\infty(t)M\Xi'W_0(t)^{-1})_- = 0$. Thus we may assume that $(W_\infty(t)M\Xi'W_0(t)^{-1})_- = 0$ for $|t| < t_0$ and $(W_\infty(t)M\Xi'W_0(t)^{-1})_- \neq 0$ at $t = t_0$. Note that the value of $t_0$ depends on the choice of $\Xi'$. Thus the Lax operator of the Toda lattice $L(t) = W_\infty(t)MW_\infty(t)^{-1}$ has singularity at $t = t_0$. Thus we may show that we can extend $L(t)$ analytically to all $t \in \mathbb{R}$ for certain $\Xi'$. Put $W_\infty(t) = (w^\infty_{i,j}(t))$. Since $L_{1,1}(t) = -w^\infty_{2,1}(t), \dots, L_{i,i}(t) = w^\infty_{i,i-1}(t) - w^\infty_{i+1,i}(t), \dots, L_{n,n}(t) = w^\infty_{n,n-1}(t)$ and $L_{i,i-1}(t) = \exp(\int^t L_{i,i}(t) - L_{i+1,i+1}(t)\,dt)$, $i = 1, \dots, n-1$, we may show that $w^\infty_{j,j-1}(t)$, $j = 2, \dots, n$, can be extended to entire functions on $\mathbb{R}$. For $\vec{x} = {}^t(x_1, \dots, x_n)$, we use a notation $\vec{x}(j) = {}^t(x_1, \dots, x_j)$. Put

$$\tau_j(t) = \big|e^{t\alpha_1}\vec{\xi}_1(j-1) + \vec{\xi}'_1(j-1), \dots, e^{t\alpha_{j-1}}\vec{\xi}_{j-1}(j-1) + \vec{\xi}'_{j-1}(j-1)\big|.$$

In (6) put $U(t)=W_\infty(t)MA'W_0(t)^{-1}$. Since $M$ is invertible, we have $A'=M^{-1}W_\infty(t)^{-1}U(t)W_0(t)$. Then we have

$$\left\{W_\infty(t)^{-1}-M^{-1}W_\infty(t)^{-1}U(t)\right\}W_0(t)=e^{tM}A. \tag{14}$$

Consider the Gauss decomposition

$$V_\infty(t)^{-1}V_0(t)=W_\infty(t)^{-1}-M^{-1}W_\infty(t)^{-1}U(t),$$

where $V_\infty(t)\in\bar{N}$ and $V_0(t)\in B$. Then we have the Gauss decomposition of $e^{tM}$ such as

$$V_\infty(t)^{-1}\left(V_0(t)W_0(t)\right)=e^{tM}A. \tag{15}$$

Since the modified Gauss decomposition of (6) with the condition

$$\left(W_\infty(t)^{-1}MA'W_0(t)^{-1}\right)_-=0$$

gives the same flow of the Gauss decomposition of (15), the Gauss decomposition of (6) on $\mathcal{O}_T$ is equivalent to the ordinary Gauss decomposition of (15). This implies that the modified Gauss decomposition (10) is equivalent to the Gauss decomposition $\tilde{W}_\infty(t)^{-1}\tilde{W}_0(t)=\Xi\,\mathrm{diag}(e^{t\alpha_1},\dots,e^{t\alpha_n})$ on the stable sector $(W_\infty(t)M\Xi'W_0(t)^{-1})_-=0$. This implies the $\tau$-function expression

$$w^{\infty}_{j,j-1}(t)=-\partial/\partial t\log\tau_j(t)+(m_{1,1}+\cdots+m_{j-1,j-1}).$$

Let us use notations such as $|1',2,\dots,j-1|:=|\vec{\xi}_1'(j-1),\vec{\xi}_2(j-1),\dots,\vec{\xi}_j(j-1)|$. Then we have

$$\tau_j(t)=\left|1',\dots,(j-1)'\right|+e^{t\alpha_1}\left|1,2',\dots,(j-1)'\right|+\cdots+e^{t\alpha_1+\cdots+t\alpha_{j-1}}|1,\dots,j-1|.$$

At first we choose $\Xi'$ so that the conditions $|1',\dots,(j-1)'|\neq 0$, $j=2,\dots,n$, are satisfied. Then we have

$$-\partial/\partial t\log\tau_j(t)=-\partial/\partial t\log\left|1+\frac{|1,2',\dots,(j-1)'|}{|1',\dots,(j-1)'|}e^{t\alpha_1}+\cdots+\frac{|1,\dots,j-1|}{|1',\dots,(j-1)'|}e^{t\alpha_1+\cdots+t\alpha_{j-1}}\right|.$$

Put

$$\begin{aligned}F_j(t)=1&+\left(\left|1,2',\dots,(j-1)'\right|/\left|1',\dots,(j-1)'\right|\right)e^{t\alpha_1}+\cdots\\&+\left(|1,\dots,j-1|/\left|1',\dots,(j-1)'\right|\right)e^{t\alpha_1+\cdots+t\alpha_{j-1}}.\end{aligned}$$

In the limit $t\to\pm\infty$, we see that $|F_j(t)|\to+\infty$ or 1. We can take $\Xi'$ so as to $|F_j(t)|>0$, $j=2,\dots,n$, for any $t\in\mathbb{R}$. For example we may take $\Xi'$ so as to $|1,\dots,\mu-1,\mu',\dots,(j-1)'|/|1',\dots,(j-1)'|>0$, $\mu=2,\dots,j-1$ and $j=2,\dots,n$. Then $\log\tau_j(t)$ and, therefore, $w_{j,j-1}(t)$, $j=2,\dots,n$, are entire functions on $\mathbb{R}$. □

The equivalence between the Gauss decompositions (6) and (15) come from the coordinate transform of the flag variety $GL_n(\mathbb{R})/B$. Since $w_{j,j-1}(t)$, $j=2,\dots,n$, are entire functions on $\mathbb{R}$, we call them stable solutions of the $A_n$-type Toda lattice.

**Example** *(1-Soliton solution).* Put $\Xi'=(\vec{\xi}',\vec{0},\dots,\vec{0})$, where $\vec{\xi}'={}^t(\xi_1',\dots,\xi_n')$. We assume the relation $(W_\infty(0)M\Xi'W_0^{-1}(0))_-=0$. Thus we have the $\tau$-function expressions in the neighborhood of $t=0$

$$w^{\infty}_{j,j-1}(t)=-\partial/\partial t\log\tau_j(t)+(m_{1,1}+\cdots+m_{j-1,j-1}),\quad j=2,\dots,n,$$

where

$$\tau_j(t)=\left|e^{t\alpha_1}\vec{\xi}_1(j-1)+\vec{\xi}'(j-1),e^{t\alpha_2}\vec{\xi}_2(j-1),\dots,e^{t\alpha_{j-1}}\vec{\xi}_{j-1}(j-1)\right|.$$

Then we have

$$\begin{aligned}w^{\infty}_{j,j-1}(t)=&-\partial/\partial t\log\left\{1+\left(|1,2,\dots,j-1|/\left|1',2,\dots,j-1\right|\right)e^{t\alpha_1}\right\}\\&+(m_{1,1}+\cdots+m_{j-1,j-1}-\alpha_2-\cdots-\alpha_{j-1}).\end{aligned} \tag{16}$$

Put

$$\Xi_{j-1,i} = (-1)^{i+1} \begin{vmatrix} \xi_{1,2} & \cdots & \xi_{1,j-1} \\ \vdots & \vdots & \vdots \\ \xi_{i-1,2} & \cdots & \xi_{i-1,j-1} \\ \xi_{i+1,2} & \cdots & \xi_{i+1,j-1} \\ \vdots & \vdots & \vdots \\ \xi_{j-1,2} & \cdots & \xi_{j-1,j-1} \end{vmatrix} \quad \text{and} \quad \Xi(j-1) = \begin{vmatrix} \xi_{1,1} & \cdots & \xi_{1,j-1} \\ \vdots & \vdots & \vdots \\ \xi_{j-1,1} & \cdots & \xi_{j-1,j-1} \end{vmatrix}.$$

For $\beta \in \mathbb{R}$, put

$$\xi_i' = \begin{vmatrix} \Xi_{1,1} & 0 & \cdots & \beta\Xi(1) & \cdots & 0 \\ \Xi_{2,1} & \Xi_{2,2} & \cdots & \beta^2\Xi(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ \Xi_{n-1,1} & \Xi_{n-1,2} & \cdots & \beta^{n-1}\Xi(n-1) & \cdots & \Xi_{n-1,n-1} \end{vmatrix} \Big/ \Xi_{1,1}\cdots\Xi_{n-1,n-1},$$
$$i = 1, \ldots, n-1,$$

where ${}^t(\beta\Xi(1), \beta^2\Xi(2), \ldots, \beta^{n-1}\Xi(n-1))$ is the $i$-th column. Then we have a 1-soliton solution such as

$$w^{\infty}_{j,j-1}(t) = -\partial/\partial t \log\bigl(1 + \beta^{-j+1}e^{t\alpha_1}\bigr) + (m_{1,1} + \cdots + m_{j-1,j-1} - \alpha_2 - \cdots - \alpha_{j-1}),$$
$$j = 2, \ldots, n. \tag{17}$$

Let us express fields of $q_j(t)$ of (1) except $q_1(t)$ and $q_n(t)$ by these $\tau$-functions. From Lax form of Toda lattice, we have

$$\dot{q}_j(t) = w^{\infty}_{j,j-1}(t) - w^{\infty}_{j+1,j}(t). \tag{18}$$

Put $\beta = e^{\kappa}$. Then we have

$$\dot{q}_j(t) = \partial/\partial t \log \frac{1 + e^{-j\kappa + t\alpha_1}}{1 + e^{-(j-1)\kappa + t\alpha_1}} + c_j, \quad 2 \leqslant j \leqslant n-1, \tag{19}$$

where $c_j$ are constants. From (19) we have

$$q_j(t) = \log \frac{1 + e^{-j\kappa + t_1\alpha}}{1 + e^{-(j-1)\kappa + t\alpha_1}} + c_j t + d_j, \quad 2 \leqslant j \leqslant n-1, \tag{20}$$

where $d_j$ are constants. The non-linear term of (20) coincides with well-known 1-soliton solution of the (infinite) Toda lattice [7]. The term of linear function of (20) is an effect of finiteness of the lattice. This linear term does not affect the stability of solution. We will deal with construction of $n$-soliton ($n \geqslant 2$) solutions from stable solutions obtained in this Note elsewhere. Note that we cannot take $\beta$ arbitrary. The parameter $\beta$ must satisfies the equation of condition of "stable sector" $(W_\infty(0)M\Xi' W_0(0)^{-1})_- = 0$ which is the analytic relations between $\beta$ and the eigenvalues $\alpha_1, \ldots, \alpha_n$ and corresponds to the dispersion relations of soliton solutions of the infinite Toda lattice.

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