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Group Theory

Characterizing finite p-groups by their Schur multipliers

Caractérisation des p-groupes finis par leurs multiplicateurs de Schur

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ABSTRACT

It has been proved in J.A. Green (1956) [5] for every p-group of order p^n, $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$, where $t(G) \geqslant 0$. In Ya.G. Berkovich (1991) [1], G. Ellis (1999) [4], and X. Zhou (1994) [14], the structure of G has been characterized for $t(G) = 0, 1, 2, 3$ by several authors. Also in A.R. Salemkar et al. (2007) [12], the structure of G characterized when $t(G) = 4$ and $Z(G)$ is elementary abelian, but there are some missing points in classifying the structure of these groups. This paper is devoted to classify the structure of G when $t(G) = 4$ without any condition and with a short and quite different way to that of Ya.G. Berkovich (1991) [1], G. Ellis (1999) [4], A.R. Salemkar et al. (2007) [12], and X. Zhou (1994) [14].

RÉSUMÉ

Il est montré dans J.A. Green (1956) [5] que pour tout p-groupe d'ordre p^n on a $|\mathcal{M}(G)| = p^{\frac{n(n-1)}{2}-t(G)}$ où $t(G) \geqslant 0$. Dans Ya.G. Berkovich (1991) [1], G. Ellis (1999) [4], et X. Zhou (1994) [14] la structure de G a été classifiée par plusieurs auteurs pour $t(G) = 0, 1, 2, 3$. Également, dans A.R. Salemkar et al. (2007) [12] la structure de G est caractérisée lorsque $t(G) = 4$ et $Z(G)$ est abelien élémentaire, mais il y a quelques trous dans la classification complète de ces groupes. Cette Note est consacrée à la caractérisation de la structure de G lorsque $t(G) = 4$, sans restriction aucune et d'une manière différente, plus directe que les approches de Ya.G. Berkovich (1991) [1], G. Ellis (1999) [4], A.R. Salemkar et al. (2007) [12], et X. Zhou (1994) [14].

1. Introduction and motivation

The literature of $\mathcal{M}(G)$, the Schur multiplier is going back to the work of Schur in 1904. It is important to know for which classes of groups the structure of group can be completely described only by the order of $\mathcal{M}(G)$. The answer to this question for the class of finite p-group, was born in a result of Green. It is shown that in [5], for a given p-group of order p^n, $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$ where $t(G) \geqslant 0$. Several authors tried to characterize the structure of G by $t(G)$. The structure of G was classified in [1,14] for $t(G) = 0, 1, 2$. When $t(G) = 3$, Ellis in [4] classified the structure of G by a different method to that of [1,14]. He also could find the same results for $t(G) = 0, 1, 2$.

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By a similar technique to [4, Theorem 1], the structure of p-groups with $t(G)=4$ has been determined in [12] when $Z(G)$ is elementary abelian, but it seems there are some missing points in classifying the structure of these groups. The Main Theorem shows that there are some groups which are not seen in these classification.

Recently in [8–10], the author gives some results on the Schur multiplier of non-abelian p-groups. Handling these results, the present paper is devoted to classify the structure of all finite p-groups when $t(G)=4$ without any condition.

2. Main results

In this section, at first we summarize some known results which are used throughout this paper and then we classify the structure of all p-groups when $t(G)=4$. Since abelian groups with the property $t(G)=4$ are determined in [12, Main Theorem (a)], we concentrate on non-abelian p-groups.

Using notations and terminology of [4], here D_8 and Q_8 denote the dihedral and quaternion group of order 8, E_1 and E_2 denote the extra special p-groups of order p^3 of exponent p and p^2, respectively. Also $\mathbb{Z}_{p^n}^{(m)}$ denotes the direct product of m copies of the cyclic group of order p^n.

In this paper, we say that G has the property $t(G)=4$ or briefly with $t(G)=4$, if $|\mathcal{M}(G)|=p^{\frac{1}{2}n(n-1)-4}$.

Theorem 2.1. *(See [8, Main Theorem].) Let G be a non-abelian p-group of order p^n. If $|G'|=p^k$, then we have*

$$\bigl|\mathcal{M}(G)\bigr| \leqslant p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$

In particular,

$$\bigl|\mathcal{M}(G)\bigr| \leqslant p^{\frac{1}{2}(n-1)(n-2)+1},$$

and the equality holds in the last bound if and only if $G=E_1\times Z$, where Z is an elementary abelian p-group.

Theorem 2.2. *(See [10, Main Theorem].) Let G be a non-abelian p-group of order p^n. Then $|\mathcal{M}(G)|=p^{\frac{1}{2}(n-1)(n-2)}$ if and only if G is isomorphic to one of the following groups.*

(i) *$D_8\times Z$, where Z is an elementary abelian p-group.*
(ii) *$\mathbb{Z}_p^{(4)}\rtimes\mathbb{Z}_p$ $(p\neq 2)$.*

Theorem 2.3. *(See [6, Theorem 2.2.10].) For every pair of finite groups H and K, we have*

$$\mathcal{M}(H\times K)\cong\mathcal{M}(H)\times\mathcal{M}(K)\times\frac{H}{H'}\otimes\frac{K}{K'}.$$

Theorem 2.4. *(See [6, Theorem 3.3.6].) Let G be an extra special p-group of order p^{2m+1}. Then*

(i) *If $m\geqslant 2$, then $|\mathcal{M}(G)|=p^{2m^2-m-1}$.*
(ii) *If $m=1$, then the order of Schur multipliers of D_8, Q_8, E_1 and E_2 are equal to 2, 1, p^2 and 1, respectively.*

Theorem 2.5. *Let G be a non-abelian p-group of order p^n and $n\geqslant 6$, then there is exactly one group with the property $t(G)=4$ which is isomorphic to $E_1\times\mathbb{Z}_p^{(3)}$.*

Proof. First assume that $|G'|=p$. By Theorem 2.1, if $|\mathcal{M}(G)|=p^{\frac{1}{2}(n-1)(n-2)+1}$, then $G\cong E_1\times Z$. One can check that by Theorems 2.3 and 2.4, $Z\cong\mathbb{Z}_p^{(3)}$. Otherwise, $|\mathcal{M}(G)|=p^{\frac{1}{2}n(n-1)-4}\leqslant p^{\frac{1}{2}(n-1)(n-2)}$ so $n\leqslant 5$.

Now assume that $|G'|=p^k$ $(k\geqslant 2)$, Theorem 2.1 implies that

$$\frac{1}{2}\bigl(n^2-n-8\bigr)\leqslant\frac{1}{2}(n+k-2)(n-k-1)+1\leqslant\frac{1}{2}n(n-3)+1,$$

and hence $n\leqslant 3$ unless $k=2$, in which case $n\leqslant 5$. $\square$

The following theorem is a consequence of Theorems 2.1 and 2.2.

Theorem 2.6. *Let G be a non-abelian p-group of order p^5 and $t(G)=4$. Then G is isomorphic to the*

$$\mathbb{Z}_p^{(4)}\rtimes\mathbb{Z}_p\quad(p\neq 2)\quad\textit{or}\quad D_8\times\mathbb{Z}_p^{(2)}.$$

Now we need only consider non-abelian groups of order p^4 with $t(G)=4$, because of Theorems 2.5 and 2.6.
In the case $p=2$, the following lemma characterizes all groups of order 16 with $t(G)=4$.

Lemma 2.7. *Let G be a non-abelian p-group of order 16 with $t(G)=4$, then G is isomorphic to one of the groups listed below*

(i) $Q_8 \times \mathbb{Z}_2$,
(ii) $\langle a, b \mid a^4 = 1, b^4 = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^2b^2 \rangle$,
(iii) $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$.

Proof. The Schur multiplier of all groups of order 16 is determined in Table I of [2] (also see [11]). □

Lemma 2.8. *Let G be a non-abelian group of order p^4 ($p \neq 2$) and $Z(G)$ be of exponent p^2 with $t(G)=4$. Then $G \cong E_4$, where E_4 is the unique central product of a cyclic group of order p^2 and a non-abelian group of order p^3.*

Proof. If G/G' is not elementary abelian, then one can check that G is of exponent p^3. Since G is the unique 2-generated p-groups of class 2 and exponent p^3, groups are listed in [7, p. 4] shows $G \cong \langle a, b \mid a^{p^3} = 1, b^p = 1, a^{p^2} = [a,b] \rangle$. Now by using [7, Theorem 49], we have $|\mathcal{M}(G)| = 1$. Thus G/G' is elementary abelian. The rest of proof is obtained directly by invoking [8, Lemma 2.1]. □

Lemma 2.9. *Let G be a group of order $p^4 (p \neq 2)$, $|G'| = p$, $Z(G)$ of exponent p and $t(G)=4$, then G is isomorphic to*

$$E_2 \times \mathbb{Z}_p \quad \text{or} \quad \langle a, b \mid a^{p^2} = 1, b^p = 1, [a,b,a] = [a,b,b] = 1 \rangle.$$

Proof. First suppose that G/G' is elementary abelian. Then [8, Lemma 2.1] and Theorem 2.4 show that $G \cong E_2 \times \mathbb{Z}_p$. Otherwise by [3, pp. 87–89], there are two groups

$$\langle a, b \mid a^{p^2} = 1, b^p = 1, [a,b,a] = [a,b,b] = 1 \rangle \quad \text{and}$$
$$\langle a, b \mid a^{p^2} = b^{p^2} = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^p \rangle$$

such that $Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, $G/G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ and $G' \cong \mathbb{Z}_p$.

Since the first has a central subgroup H such that $G/H \cong E_1$ [6, Corollary 2.5.3(i)] shows $p^2 \leqslant |\mathcal{M}(G)|$. Now by taking $B = G'$ in [6, Theorem 2.5.5(i)], we have $|\mathcal{M}(G)| \geqslant p^2$, and so $|\mathcal{M}(G)| = p^2$. On the other hand, [6, Theorem 2.2.5] shows that the second group has $|\mathcal{M}(G)| = p$, from which the result follows. □

Lemma 2.10. *Let G be a group of order p^4 ($p \neq 2$), $|G'| = p^2$ and $t(G)=4$. Then G is isomorphic to one of the following groups.*

(i) $\langle a, b \mid a^9 = b^3 = 1, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1 \rangle$,
(ii) $\langle a, b \mid a^p = 1, b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1 \rangle$ ($p \neq 3$).

Proof. The fifteen groups of odd order p^4 are listed in [3] or [13]. Our conditions reduce these groups to the unique group (see also [4, p. 4177] for more details). □

In the following theorem we summarize the results.

Theorem 2.11. *Let G be a non-abelian group of order p^n with $t(G)=4$, then G is isomorphic to one of the following groups.*
For $p=2$,

(1) $D_8 \times \mathbb{Z}_p^{(2)}$,
(2) $Q_8 \times \mathbb{Z}_2$,
(3) $\langle a, b \mid a^4 = 1, b^4 = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^2b^2 \rangle$,
(4) $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$.

For $p \neq 2$,

(5) E_4,
(6) $E_1 \times \mathbb{Z}_p^{(3)}$,
(7) $\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$,

(8) $E_2 \times \mathbb{Z}_p$,

(9) $\langle a, b \mid a^{p^2} = 1, b^p = 1, [a, b, a] = [a, b, b] = 1\rangle$,

(10) $\langle a, b \mid a^9 = b^3 = 1, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1\rangle$,

(11) $\langle a, b \mid a^p = 1, b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1\rangle$ $(p \neq 3)$.

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