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Partial Differential Equations

An inverse problem for a time-dependent Schrödinger operator in an unbounded strip

Un problème inverse pour un opérateur de Schrödinger dépendant du temps dans une bande non bornée

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A B S T R A C T

In this Note we prove a stability result for two independent coefficients (each one depending on only one space variable and the potential also depending on the time variable) for a time-dependent Schrödinger operator in an unbounded strip with one observation on an unbounded subset of the boundary and the data of the solution at a fixed time on the whole domain.

R É S U M É

Dans cette Note, on prouve un résultat de stabilité pour deux coefficients indépendants (chacun d'eux dépendant d'une seule variable d'espace et le potentiel dépendant aussi de la variable temps) pour un opérateur de Schrödinger avec une observation sur une partie non bornée du bord et la donnée de la solution à un temps fixé sur tout le domaine.

Version française abrégée

Considérons l'équation de Schrödinger (1). L'objet de cette note est d'obtenir un résultat de stabilité pour les deux coefficients indépendants $a = a(x_2)$ et $b = f(t)g(x_1)$ avec une observation sur une partie non bornée du bord. On considèrera deux cas pour le potentiel b : l'une des deux fonctions f ou g sera connue. La méthode employée est basée sur une estimation de Carleman globale (2) avec une seule observation frontière. Puis nous rappelons une estimation d'énergie (3) et une estimation de type Carleman (4) pour un opérateur différentiel du premier ordre.

Nous prouvons ensuite le résultat principal de stabilité Lipschitz pour les deux coefficients a et b avec une observation et la donnée de la solution sur tout le domaine en $t = 0$ et en supposant connus ces deux coefficients sur le bord :

La norme H^1 du coefficient de diffusion a et la norme L^2 du potentiel b ainsi que de ses dérivées en temps est estimée par la norme L^2 de la dérivée normale de la solution ainsi que de sa dérivée première ou seconde en temps, sur une partie non bornée du bord (*voir* (8), (9)).

Nous suivons la méthode développée dans [2] pour établir ce résultat de stabilité.

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1. Introduction

This paper is an improvement of the work [3] in the sense that we determine two independent coefficients: the diffusion coefficient $a := a(x_2)$ and the potential $b := b(x_1, t) = f(t)g(x_1)$, but one of them depending on the time variable, with one observation and the data of the solution at a fixed time (as in [4]).

Let $\Omega = \mathbb{R} \times (0, d)$ be an unbounded strip of $\mathbb{R}^2$ with a fixed width d. Let ν be the outward unit normal to Ω on $\Gamma = \partial\Omega$. We denote $x = (x_1, x_2)$ and $\Gamma = \Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = \{x \in \Gamma;\ x_2 = d\}$ and $\Gamma^- = \{x \in \Gamma;\ x_2 = 0\}$. We consider the following Schrödinger equation

$$\begin{cases} Hq(x,t) := i\partial_t q(x,t) + \nabla \cdot \big(a(x_2)\nabla q(x,t)\big) + b(x_1,t)q(x,t) = 0 & \text{in } Q = \Omega \times (0,T), \\ q(x,t) = F(x,t) & \text{on } \Sigma = \partial\Omega \times (0,T), \\ q(x,0) = q_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

where $a \in \mathcal{C}^3(\overline{\Omega})$, $b \in \mathcal{C}^2(\overline{Q})$ and $a(x) \geqslant a_{min} > 0$. Moreover, we assume that a (resp. b) and all its derivatives up to order three (resp. two) are bounded.

Our problem can be stated as follows:
Is it possible to determine the coefficients a and b from the measurement of $\partial_\nu(q)$, $\partial_\nu(\partial_t q)$ and $\partial_\nu(\partial_t^2 q)$ on Γ^+?

We will consider two cases for the potential $b := b(x_1, t) = f(t)g(x_1)$.

In a first case we consider q (resp. $\tilde{q}$) a solution of (1) associated with $(a,\ f,\ g,\ F,\ q_0)$ (resp. $(\tilde{a},\ \tilde{f},\ g,\ F,\ q_0)$) satisfying some regularity properties:

Assumption 1.1.

- $\tilde{q}$ and all its derivatives up to order four are bounded.
- q_0 is a real valued function in $\mathcal{C}^3(\overline{\Omega})$.
- q_0 and all its derivatives up to order three are bounded.

Our main result is: if $(f - \tilde{f})(0) \neq 0$ then

$$\|a - \tilde{a}\|^2_{H^1(\Omega)} + \sum_{i=0}^{1} \big\|\partial_t^i (b - \tilde{b})\big\|^2_{L^2(\Omega \times (-T,T))} \leqslant C \sum_{i=0}^{1} \big\|\partial_\nu\big(\partial_t^i (q - \tilde{q})\big)\big\|^2_{L^2(\Gamma^+ \times (-T,T))}$$

and if $(f - \tilde{f})(0) = 0$, $(f - \tilde{f})'(0) \neq 0$, then

$$\|a - \tilde{a}\|^2_{H^1(\Omega)} + \sum_{i=0}^{2} \big\|\partial_t^i (b - \tilde{b})\big\|^2_{L^2(\Omega \times (-T,T))} \leqslant C \left[\sum_{i=0}^{2} \big\|\partial_\nu\big(\partial_t^i (q - \tilde{q})\big)\big\|^2_{L^2(\Gamma^+ \times (-T,T))} + \big\|\partial_t (q - \tilde{q})(\cdot, 0)\big\|^2_{H^3(\Omega)} \right]$$

where $b = fg$, $\tilde{b} = \tilde{f}g$, C is a positive constant which depends on (Ω, Γ, T) and where the above norms are weighted Sobolev norms ($\partial_t^i q$ denoting the ith-time derivatives of q).

In a second case we consider q (resp. $\tilde{q}$) a solution of (1) associated with $(a,\ g,\ f,\ F,\ q_0)$ (resp. $(\tilde{a},\ \tilde{g},\ f,\ F,\ q_0)$) satisfying the regularity properties (1.1). We obtain the same results as above by considering the case $f(0) \neq 0$ on one hand and the case $f(0) = 0$, $f'(0) \neq 0$ on the other hand, with this time $b = fg$, $\tilde{b} = f\tilde{g}$.

The major novelty of this paper is to consider the coefficient b depending on the time variable. In several works, the problem of the identification of coefficients for the Schrödinger operator have been studied (see [1] in bounded domains and [2–4] in unbounded domains) but in all of these works, the coefficients only depended on the space variable and not on the time variable. Note also that in [5], for a magnetic Schrödinger operator with a time-dependent magnetic potential $\chi(t)a$, there is a result for the identification of the coefficient a which does not depend on the time variable. For the problem of the identification of a potential $b(x,t)$, to our knowledge, there is no result.

We will use the global Carleman estimate given in [2–4] adapted to our case. We also need an energy estimate for the operator H given in [2] and a Carleman type estimate for a first order differential equation proved in [6] for bounded domains and in [2] for unbounded domains. Using all these previous estimates, we give our stability results for the coefficients a and b.

This Note is organized as follows. In Section 2, we recall an adapted global Carleman estimate for the operator H (see [2–4]) and also an energy estimate (see [2,3]). In Section 3 we prove our two main results, a stability result for the coefficients a and f on the one hand and a stability result for the coefficients a and g on the other hand.

2. Global Carleman inequality and energy estimate

Consider the set $\Lambda(R_1) := \{f \in L^\infty(\overline{Q}),\ \|f\|_{L^\infty(\overline{Q})} < R_1\}$ with R_1 be a strictly positive and fixed real. Let a be a real-valued function in $\mathcal{C}^3(\overline{\Omega})$ and b be a real-valued function in $\mathcal{C}^2(\overline{Q})$ such that

Assumption 2.1.

- $a \geqslant a_{min} > 0$, a and all its derivatives up to order three are in $\Lambda(R_1)$,
- b and all its derivatives up to order two are in $\Lambda(R_1)$.

Let $\tilde{\beta}$ be a $\mathcal{C}^4(\overline{\Omega})$ positive function such that there exist positive constants C_0, C_{pc} which satisfy

Assumption 2.2.

- $|\nabla\tilde{\beta}| \geqslant C_0 > 0$ in $\overline{\Omega}$, $\partial_\nu\tilde{\beta} \leqslant 0$ on Γ^-,
- $\tilde{\beta}$ and all its derivatives up to order four are in $\Lambda(R_1)$,
- $2\Re(D^2\tilde{\beta}(\zeta,\bar{\zeta})) - a\nabla a \cdot \nabla\tilde{\beta}|\zeta|^2 + 2a^2|\nabla\tilde{\beta}\cdot\zeta|^2 \geqslant C_{pc}|\zeta|^2$, for all $\zeta \in \mathbb{C}$ where $D^2\tilde{\beta} = \begin{pmatrix} a\partial_{x_1}(a\partial_{x_1}\tilde{\beta}) & a\partial_{x_1}(a\partial_{x_2}\tilde{\beta}) \\ a\partial_{x_2}(a\partial_{x_1}\tilde{\beta}) & a\partial_{x_2}(a\partial_{x_2}\tilde{\beta}) \end{pmatrix}$.

This assumption imposes restrictive conditions for the choice of the functions a in connection with the function $\tilde{\beta}$. See [3] for a class of coefficients a which satisfy Assumption 2.2. Indeed, note that here a only depends on x_2 and if we consider $a \in \{f \in C^1(\Omega); \exists r_0$ positive constant, $-f\partial_{x_2}f\partial_{x_2}\tilde{\beta} \geqslant r_0 > 0,\ f\partial_{x_2}f\partial_{x_2}\tilde{\beta} + 2f^2(\partial^2_{x_2}\tilde{\beta} + (\partial_{x_2}\tilde{\beta})^2) \geqslant r_0 > 0\}$ then a function $\tilde{\beta}(x_1,x_2) = \tilde{\beta}(x_2)$ is available (for example, $a(x) = e^{-x_2}$ and $\tilde{\beta}(x) = e^{x_2}$). Then, we define $\beta = \tilde{\beta} + K$ with $K = m\|\tilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (-T,T)$, we define the following weight functions $\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{(T+t)(T-t)}$, $\eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)}$. Let H be the operator defined by $Hq := i\partial_t q + \nabla\cdot(a\nabla q) + bq$ in $\widetilde{Q} = \Omega \times (-T,T)$. We assume that Assumptions 2.1 and 2.2 are satisfied in this paper. Then the following global Carleman estimates holds.

Theorem 2.3. *Then there exist $\lambda_0 > 0$, $s_0 > 0$ and a positive constant $C = C(\Omega,\Gamma,T,C_0,C_{pc},R_1)$ such that, for any $\lambda \geqslant \lambda_0$ and any $s \geqslant s_0$, the next inequality holds*:

$$s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} e^{-2s\eta}\varphi^3|q|^2 + s\lambda \int_{-T}^{T}\int_{\Omega} e^{-2s\eta}\varphi|\nabla q|^2 \leqslant C\left[s\lambda \int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi|\partial_\nu q|^2\,\partial_\nu\beta\,\mathrm{d}\sigma\,\mathrm{d}t + \int_{-T}^{T}\int_{\Omega} e^{-2s\eta}|Hq|^2\right] \tag{2}$$

for all q satisfying $Hq \in L^2(\Omega\times(-T,T))$, $q \in L^2(-T,T;H^1_0(\Omega))$, $\partial_\nu q \in L^2(-T,T;L^2(\Gamma))$.

We recall now an energy estimate for q with a single observation acting on the upper part Γ^+ of the boundary Γ in the right-hand side of the estimate (see [2,3]). Here again, as for the global Carleman estimate, even if the potential b do not depend on the time variable in [2,3], we can exactly proceed as in these previous papers. We denote by $E(t) := \int_\Omega e^{-2s\eta(x,t)}|q(x,t)|^2\,\mathrm{d}x + \int_\Omega a\varphi^{-1}(x,t)e^{-2s\eta(x,t)}|\nabla q(x,t)|^2\,\mathrm{d}x$, where $\varphi^{-1} = \frac{1}{\varphi}$. We give an estimate for the energy $E(0)$ in Theorem 2.4.

Theorem 2.4. *Let q be a function equals to zero on $\partial\Omega\times(-T,T)$ in the following class: $q \in C([0,T],H^1(\Omega))$, $\partial_\nu q \in L^2(0,T,L^2(\Gamma))$. Then there exists a positive constant $C = C(\Omega,\Gamma,T) > 0$ such that for s and λ sufficiently large*

$$E(0) \leqslant C\left[s^2\lambda^2 \int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi\partial_\nu\beta|\partial_\nu v|^2\,\mathrm{d}\sigma\,\mathrm{d}t + s\lambda\iint_Q e^{-2s\eta}|Hq|^2\right]. \tag{3}$$

We also recall a useful lemma (see [6,2]) for the following first order differential operator.

Lemma 2.5. *Let P the operator defined by $Pg := Q_0\cdot\nabla g$ with $Q_0 = (q_0, \partial_{x_2}q_0)$ which satisfies $|Q_0\cdot\nabla\beta| \geqslant cst > 0$. Denote by $\eta_0(x) := \eta(x,0)$ and $\varphi_0(x) := \varphi(x,0)$. Then there exist positive constants $\lambda_1 > 0$, $s_1 > 0$ and $C = C(\Omega,\Gamma,T)$ such that for all $\lambda \geqslant \lambda_1$, $s \geqslant s_1$, and for all $g \in H^1_0(\Omega)$,*

$$s^2\lambda^2 \int_\Omega \varphi_0 e^{-2s\eta_0}|g|^2 \leqslant C\int_\Omega \varphi_0^{-1}e^{-2s\eta_0}|Pg|^2. \tag{4}$$

3. Stability result

Let q and $\tilde{q}$ be solutions of

$$\begin{cases} i\partial_t q + \nabla\cdot(a\nabla q) + bq = 0 & \text{in } \Omega\times(0,T),\\ q(x,t) = F(x,t) & \text{on } \partial\Omega\times(0,T),\\ q(x,0) = q_0(x) & \text{in } \Omega,\end{cases}\qquad \begin{cases} i\partial_t \tilde{q} + \nabla\cdot(\tilde{a}\nabla \tilde{q}) + \tilde{b}\tilde{q} = 0 & \text{in } \Omega\times(0,T),\\ \tilde{q}(x,t) = F(x,t) & \text{on } \partial\Omega\times(0,T),\\ \tilde{q}(x,0) = q_0(x) & \text{in } \Omega,\end{cases} \tag{5}$$

where a, b, $\tilde{a}$ and $\tilde{b}$ satisfy Assumption 2.1. We suppose that Assumption 1.1 is checked, then we extend the functions q (resp. b) on $\widetilde{Q} = \Omega\times(-T,T)$ by the formula $q(x,t) = \bar{q}(x,-t)$ (resp. $b(x,t) = b(x,-t)$) for every $(x,t)\in\Omega\times(-T,0)$. Denote by $\Sigma_T := \partial\Omega\times(-T,T)$. If we set $u = q-\tilde{q}$, $v = \partial_t u$, $w = \partial_t v$, $\alpha = \tilde{a}-a$, $\gamma = \tilde{b}-b$, $b_0(x) = b(x,0)$, $\tilde{b}_0(x) = \tilde{b}(x,0)$, $b_1(x) = \partial_t b(x,0)$, $\tilde{b}_1(x) = \partial_t\tilde{b}(x,0)$, $\gamma_0(x) = \gamma(x,0)$, $\gamma_1(x) = \partial_t\gamma(x,0)$, $q_1(x) = \partial_t\tilde{q}(x,0)$ then u, v and w satisfy

$$\begin{cases} i\partial_t u + \nabla\cdot(a\nabla u) + bu = \nabla\cdot(\alpha\nabla\tilde{q}) + \gamma\tilde{q} & \text{in } \widetilde{Q},\\ u(x,t) = 0 & \text{on } \Sigma_T,\\ u(x,0) = 0 & \text{in } \Omega,\end{cases}\qquad \begin{cases} i\partial_t v + \nabla\cdot(a\nabla v) + bv = V & \text{in } \widetilde{Q},\\ v(x,t) = 0 & \text{on } \Sigma_T,\\ v(x,0) = \frac{1}{i}\big(\nabla\cdot(\alpha\nabla q_0) + \gamma_0 q_0\big) & \text{in } \Omega,\end{cases} \tag{6}$$

with $V := -\partial_t b\, u + \nabla\cdot(\alpha\nabla(\partial_t\tilde{q})) + \gamma\,\partial_t\tilde{q} + \partial_t\gamma\,\tilde{q}$.

$$\begin{cases} i\partial_t w + \nabla\cdot(a\nabla w) + bw = -2\partial_t b v - \partial_t^2 b u + \nabla\cdot\big(\alpha\nabla\big(\partial_t^2\tilde{q}\big)\big) + 2\partial_t\gamma\,\partial_t\tilde{q} + \partial_t^2\gamma\tilde{q} + \gamma\,\partial_t^2\tilde{q} & \text{in } \widetilde{Q},\\ w(x,t) = 0 & \text{on } \Sigma_T,\\ w(x,0) = \frac{1}{i}\big(\nabla\cdot(\alpha\nabla q_1) + \gamma_0 q_1 + \gamma_1 q_0 - b_1 u(x,0) - b_0 v(x,0) - \nabla\cdot\big(a\nabla v(x,0)\big)\big) & \text{in } \Omega.\end{cases} \tag{7}$$

First we consider the case where q and $\tilde{q}$ are solutions of (5) respectively associated with (a, f, g, F, q_0) and $(\tilde{a}, \tilde{f}, g, F, q_0)$. We recall that $b = fg$ and $\tilde{b} = \tilde{f}g$. Our main stability result expresses a perturbation result around the known solution $\tilde{q}$. It is the following one

Theorem 3.1. *Let q and $\tilde{q}$ be solutions of (5), respectively associated with (a, f, g, F, q_0) and $(\tilde{a}, \tilde{f}, g, F, q_0)$, such that $a-\tilde{a}\in H_0^1(\Omega)$, $\partial_{x_2}(a-\tilde{a})\in H_0^1(\Omega)$, $b_0-\tilde{b}_0\in H_0^1(\Omega)$ and $b_1-\tilde{b}_1\in H_0^1(\Omega)$. We assume that Assumptions 2.1, 2.2 and 1.1 are satisfied. Assume also that $Q_0 = (q_0, \partial_{x_2}q_0)$ satisfies $|Q_0\cdot\nabla\beta| \geqslant cst > 0$ and that $f, g\in\Lambda(R_1)$, $\tilde{f}\in\Lambda(R_1)$.*

If $(f-\tilde{f})(0)\neq 0$, then there exists a positive constant $C = C(\Omega,\Gamma,T,R_1)$ such that for s and λ large enough,

$$\int_\Omega e^{-2s\eta_0}\big(|a-\tilde{a}|^2 + \big|\nabla(a-\tilde{a})\big|^2\big) + \int_{-T}^{T}\int_\Omega e^{-2s\eta}\sum_{i=0}^{1}\big|\partial_t^i(b-\tilde{b})\big|^2 \leqslant C\int_{-T}^{T}\int_{\Gamma^+}\varphi e^{-2s\eta}\partial_\nu\beta\sum_{i=0}^{1}\big|\partial_\nu\big(\partial_t^i(q-\tilde{q})\big)\big|^2. \tag{8}$$

If $(f-\tilde{f})(0) = 0$ and $(f-\tilde{f})'(0)\neq 0$, then there exists a positive constant $C = C(\Omega,\Gamma,T,R_1)$ such that for s and λ large enough,

$$\int_\Omega e^{-2s\eta_0}\big(|a-\tilde{a}|^2 + \big|\nabla(a-\tilde{a})\big|^2\big) + \int_{-T}^{T}\int_\Omega e^{-2s\eta}\sum_{i=0}^{2}\big|\partial_t^i(b-\tilde{b})\big|^2$$
$$\leqslant C\int_{-T}^{T}\int_{\Gamma^+}\varphi e^{-2s\eta}\partial_\nu\beta\sum_{i=0}^{2}\big|\partial_\nu\big(\partial_t^i(q-\tilde{q})\big)\big|^2 + C\int_\Omega\varphi_0^{-1}e^{-2s\eta_0}\big[\big|\nabla\partial_{x_1}\partial_t(q-\tilde{q})(\cdot,0)\big|^2 + \big|\Delta\partial_{x_1}\partial_t(q-\tilde{q})(\cdot,0)\big|^2\big]. \tag{9}$$

Proof. We apply (4) given in Lemma 2.5, to the first order partial differential equations satisfied by α and γ_0 given by the initial condition and the derivatives of this initial condition for v in (6). Then from (3) applied for v and since $e^{-2s\eta}\leqslant e^{-2s\eta_0}$ we get for s and λ large enough

$$s^2\lambda^2\int_\Omega\varphi_0 e^{-2s\eta_0}\big(|\alpha|^2 + |\nabla\alpha|^2 + |\gamma_0|^2\big) \leqslant Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\phi e^{-2s\eta}\partial\nu\beta|\partial_\nu v|^2 + Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}\big[|u|^2 + |\gamma|^2 + |\partial_t\gamma|^2\big].$$

Using now the Carleman inequality for u we get

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}\big(|\alpha|^2+|\nabla\alpha|^2+|\gamma_0|^2\big)\leqslant Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\varphi e^{-2s\eta}\partial\nu\beta\big(|\partial_\nu v|^2+|\partial_\nu u|^2\big)$$
$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}\big[|\gamma|^2+|\partial_t\gamma|^2\big].$$

We now consider the first case $(f-\tilde f)(0)\neq 0$.

Since $t\to|\frac{(f-\tilde f)(t)}{(f-\tilde f)(0)}|$ is bounded on $[-T,T]$ we deduce that there exists a positive constant $C=C(T,R_1)$ such that for all x,t, $|\gamma(x,t)|=|(\tilde f(t)-f(t))g(x_1)|\leqslant C|\gamma(x,0)|=C|(\tilde f(0)-f(0))g(x_1)|$. Similarly we have $|\partial_t\gamma(x,t)|=|(\tilde f'(t)-f'(t))g(x_1)|\leqslant C|\gamma(x,0)|$. So we get for s and λ large enough

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}\big(|\alpha|^2+|\nabla\alpha|^2\big)+s^2\lambda^2\int_{-T}^T\int_\Omega e^{-2s\eta}\big(|\gamma|^2+|\partial_t\gamma|^2\big)\leqslant Cs^2\lambda^2\int_{-T}^T\int_{\Gamma^+}\varphi e^{-2s\eta}\partial_\nu\beta\big[|\partial_\nu v|^2+|\partial_\nu u|^2\big]$$

and we obtain (8). □

We then consider the case $(f-\tilde f)(0)=0$ and $(f-\tilde f)'(0)\neq 0$. Note that $\gamma_0=0$ and $\gamma_1(x)=(\tilde f'(0)-f'(0))g(x_1)$. Applying (4) given in Lemma 2.5 to the first order partial differential equations satisfied by γ_1 given by the derivative of the initial condition in (7), we have for s and λ large enough:

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}|\gamma_1|^2\leqslant C\int_\Omega \varphi_0^{-1}e^{-2s\eta_0}\big(\big|\partial_{x_1}w(x,0)\big|^2+|\alpha|^2+|\nabla\alpha|^2$$
$$+\big|u(x,0)\big|^2+\big|\nabla u(x,0)\big|^2+\big|v(x,0)\big|^2+\big|\nabla v(x,0)\big|^2+\big|\nabla\partial_{x_1}v(\cdot,0)\big|^2+\big|\Delta\partial_{x_1}v(\cdot,0)\big|^2\big).$$

Then using (3) for u,v,w defined by (6), (7) and the Carleman estimate applied for u and v we get

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}\big(|\alpha|^2+|\nabla\alpha|^2+|\gamma_1|^2\big)\leqslant Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\phi e^{-2s\eta}\partial\nu\beta\big(|\partial_\nu v|^2+|\partial_\nu u|^2+|\partial_\nu w|^2\big)$$
$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}\big[|\gamma|^2+|\partial_t\gamma|^2+\big|\partial_t^2\gamma\big|^2\big]$$
$$+C\int_\Omega \varphi_0^{-1}e^{-2s\eta_0}\big(\big|\nabla\partial_{x_1}v(\cdot,0)\big|^2+\big|\Delta\partial_{x_1}v(\cdot,0)\big|^2\big).$$

Using the same argument as in the first case we deduce that there exists a positive constant $C=C(T,R_1)$ such that $|\gamma|\leqslant C|\gamma_1|$, $|\partial_t\gamma|\leqslant C|\gamma_1|$ and $|\partial_t^2\gamma|\leqslant C|\gamma_1|$. We conclude as in the first case and we get (9).

Note that the assumption $|Q_0\cdot\nabla\beta|\geqslant cst>0$ imposes a restrictive condition on the initial data q_0 but if $\tilde\beta(x):=\tilde\beta(x_2)$ this condition becomes $|\partial_{x_2}q_0\partial_{x_2}\beta|\geqslant cst>0$ (for example if $\tilde\beta(x)=e^{x_2}$ then this last condition imposes that $|\partial_{x_2}q_0|\geqslant cst>0$). By the same way, if we now consider the case where q and $\tilde q$ are solutions of (5) respectively associated with (a, g, f, F, q_0) and ($\tilde a$, $\tilde g$, f, F, q_0), the stability result is (with $b=fg$ and $\tilde b=f\tilde g$).

Theorem 3.2. *Let q and $\tilde q$ be solutions of (5), respectively associated with (a,g,f,F,q_0) and $(\tilde a,\tilde g,f,F,q_0)$, such that $a-\tilde a\in H_0^1(\Omega)$, $\partial_{x_2}(a-\tilde a)\in H_0^1(\Omega)$, $b_0-\tilde b_0\in H_0^1(\Omega)$ and $b_1-\tilde b_1\in H_0^1(\Omega)$. We assume that Assumptions 2.1, 2.2 and 1.1 are satisfied. Assume also that $Q_0=(q_0,\partial_{x_2}q_0)$ satisfies $|Q_0\cdot\nabla\beta|\geqslant cst>0$ and that $f,g\in\Lambda(R_1)$, $\tilde g\in\Lambda(R_1)$. If $f(0)\neq 0$, then there exists a positive constant $C=C(\Omega,\Gamma,T,R_1)$ such that for s and λ large enough (8) is satisfied. If $f(0)=0$ and $f'(0)\neq 0$, then there exists a positive constant $C=C(\Omega,\Gamma,T,R_1)$ such that for s and λ large enough (9) is satisfied.*

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