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Cuspidal representations of $GL(n, F)$ distinguished by a maximal Levi subgroup, with F a non-archimedean local field

Représentations cuspidales de $GL(n, F)$ distinguées par un sous-groupe de Levi maximal, pour F un corps local non archimédien

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A B S T R A C T

Let ρ be a cuspidal representation of $GL(n, F)$, with F a non-archimedean local field, and H a maximal Levi subgroup of $GL(n, F)$. We show that if ρ is H-distinguished, then n is even, and H is isomorphic to $GL(n/2, F) \times GL(n/2, F)$.

R É S U M É

Soit ρ une représentation cuspidale de $GL(n, F)$, lorsque F est un corps local non archimédien, et H un sous-groupe de Levi maximal de $GL(n, F)$. Nous démontrons que si ρ est distinguée par H, alors n est pair, et H est isomorphe à $GL(n/2, F) \times GL(n/2, F)$.

1. Introduction

We recall that a complex representation (ρ, V) of a group G, with subgroup H, is said to be H-distinguished if there is a nonzero linear form L on V which is fixed by H (i.e., $L \circ \rho(h) = L$ for all h in H). Let F be non-archimedean local field, cuspidal representations of $GL(2n, F)$ distinguished by a maximal Levi subgroup $GL(n, F) \times GL(n, F)$, have been studied in several papers, see for example [3], or [2]. But this question has not been tackled for other maximal Levi subgroups, which is natural because in this case there are no distinguished cuspidal representations. As this result doesn't seem to appear in the literature yet (except for the case of maximal Levi subgroups with one factor isomorphic to $GL(1, F)$), we give a proof of it in this note.

2. Preliminaries

We denote $GL(n, F)$ by G_n for $n \geqslant 1$, and by N_n the unipotent radical of the Borel subgroup of G_n given by upper triangular matrices. For $n \geqslant 2$ we denote by U_n the group of matrices $u(x) = \begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix}$ for x in F^{n-1}.

For $n > 1$, the map $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$ is an embedding of the group G_{n-1} in G_n, we denote by P_n the subgroup $G_{n-1}U_n$ of G_n.

We fix a nontrivial character θ of $(F, +)$, and denote by θ again the character $n \mapsto \theta(\sum_{i=1}^{n-1} n_{i,i+1})$ of N_n. The normalizer of $\theta_{|U_n}$ in G_{n-1} is then P_{n-1}.

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When G is an l-group (locally compact totally disconnected group), we denote by $Alg(G)$ the category of smooth complex G-modules. If (π, V) belongs to $Alg(G)$, H is a closed subgroup of G, and χ is a character of H, we denote by δ_H the positive character of $N_G(H)$ such that if μ is a right Haar measure on H, and int is the action of $N_G(H)$ given by $(int(n)f)(h) = f(n^{-1}hn)$, on the space of smooth functions f with compact support on H, then $\mu \circ int(n) = \delta_H(n)\mu$ for n in $N_G(H)$.

If H is a closed subgroup of an l-group G, and (ρ, W) belongs to $Alg(H)$, we define the object $(ind_H^G(\rho), V_c = ind_H^G(W))$ as follows. The space V_c is the space of smooth functions from G to W, fixed under right translation by the elements of a compact open subgroup U_f of G, satisfying $f(hg) = \rho(h)f(g)$ for all h in H and g in G, and with support compact mod H. The action of G is by right translation on the functions.

If f is a function from G to another set, and g belongs to G, we will denote $L(g)f : x \mapsto f(g^{-1}x)$ and $R(g)f : x \mapsto f(xg)$.

We denote by $\mathbf{1}_H$ the trivial character of H (we simply write $\mathbf{1}$ if $H = G_0$ is the trivial group). We say that a representation π of G is H-distinguished, if the complex vector space $Hom_H(\pi, \mathbf{1}_H)$ is nonzero.

We will use the following functors following [1]:

– The functor Φ^+ from $Alg(P_{k-1})$ to $Alg(P_k)$ such that, for π in $Alg(P_{k-1})$, one has $\Phi^+\pi = ind_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta)$.
– The functor Ψ^+ from $Alg(G_{k-1})$ to $Alg(P_k)$, such that for π in $Alg(G_{k-1})$, one has $\Psi^+\pi = ind_{G_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \mathbf{1}_{U_n}) = \delta_{U_k}^{1/2}\pi \otimes \mathbf{1}_{U_n}$.

We recall the following proposition, which is a consequence of Section 3.5 and Theorem 4.4 of [1].

Proposition 2.1. *Let π be a cuspidal representation of G_n, then the restriction $\pi_{|P_n}$ is isomorphic to $(\Phi^+)^{n-1}\Psi^+(\mathbf{1})$.*

3. The result

Suppose $n = p + q$, with $p \geqslant q \geqslant 1$, we denote by $M_{(p,q)}$ the standard Levi of G_n given by matrices $\begin{pmatrix} h_p & \\ & h_q \end{pmatrix}$ with $h_p \in G_p$ and $h_q \in G_q$, and by $M_{(p,q-1)}$ the standard Levi of G_{n-1} given by matrices $\begin{pmatrix} h_p & \\ & h_{q-1} \end{pmatrix}$ with $h_p \in G_p$ and $h_{q-1} \in G_{q-1}$. We denote by $M_{(p-1,q-1)}$ the standard Levi of G_{n-2} given by matrices $\begin{pmatrix} h_{p-1} & \\ & h_{q-1} \end{pmatrix}$ with $h_{p-1} \in G_{p-1}$ and $h_{q-1} \in G_{q-1}$.

Let $w_{p,q}$ be the permutation matrix of G_n corresponding to the permutation

$$\begin{pmatrix} 1 & \ldots & p-q & p-q+1 & p-q+2 & \ldots & p-1 & p & p+1 & \ldots & p+q-2 & p+q-1 & p+q \\ 1 & \ldots & p-q & p-q+1 & p-q+3 & \ldots & p+q-3 & p+q-1 & p-q+2 & \ldots & p+q-4 & p+q-2 & p+q \end{pmatrix}.$$

Let $w_{p,q-1}$ be the permutation matrix of G_{n-1} corresponding to the permutation $w_{p,q}$ restricted to $\{1, \ldots, n-1\}$:

$$\begin{pmatrix} 1 & \ldots & p-q & p-q+1 & p-q+2 & \ldots & p-1 & p & p+1 & \ldots & p+q-2 & p+q-1 \\ 1 & \ldots & p-q & p-q+1 & p-q+3 & \ldots & p+q-3 & p+q-1 & p-q+2 & \ldots & p+q-4 & p+q-2 \end{pmatrix}.$$

Let $w_{p-1,q-1}$ be the permutation matrix of G_{n-2} corresponding to the permutation

$$\begin{pmatrix} 1 & \ldots & p-q & p-q+1 & p-q+2 & \ldots & p-2 & p-1 & p & \ldots & p+q-3 & p+q-2 \\ 1 & \ldots & p-q & p-q+1 & p-q+3 & \ldots & p+q-5 & p+q-3 & p-q+2 & \ldots & p+q-4 & p+q-2 \end{pmatrix}.$$

We denote by $H_{p,q}$ the subgroup $w_{p,q}M_{(p,q)}w_{p,q}^{-1}$ of G_n, by $H_{p,q-1}$ the subgroup $w_{p,q-1}M_{(p,q-1)}w_{p,q-1}^{-1}$ of G_{n-1}, and by $H_{p-1,q-1}$ the subgroup $w_{p-1,q-1}M_{(p-1,q-1)}w_{p-1,q-1}^{-1}$ of G_{n-2}.

The two following lemmas and propositions are a straightforward adaptation of Lemma 1 and Proposition 1 of [4].

Lemma 3.1. *Let $S_{p,q} = \{g \in G_{n-1}, \forall u \in U_n \cap H_{p,q}, \theta(gug^{-1}) = 1\}$. Then $S_{p,q} = P_{n-1}H_{p,q-1}$.*

Proof. Denoting by $L_{n-1}(g)$ the bottom row of g, one has $\theta(gu(x)g^{-1}) = \theta(L_{n-1}(g).x)$ for $u(x)$ in U_n. Hence $\theta(gug^{-1}) = 1$ for all u in $U_n \cap H_{p,q}$ if and only if $g_{n-1,j} = 0$ for $j = p-q, p-q+2, \ldots, p+q-2$. It is equivalent to say that g belongs to $P_{n-1}H_{p,q-1}$. $\square$

Lemma 3.2. *Let $S_{p,q-1} = \{g \in G_{n-2}, \forall u \in U_{n-1} \cap H_{p,q-1},\ \theta(g^{-1}ug) = 1\}$. Then $S_{p,q} = P_{n-2}H_{p-1,q-1}$.*

Proof. Denoting by $L_{n-2}(g)$ the bottom row of g, and by $u(x)$ the matrix $\begin{pmatrix} I_{n-2} & x \\ 0 & 1 \end{pmatrix}$, so that $\theta(gug^{-1}) = \theta(L_{n-2}(g).x)$. Hence $\theta(gug^{-1}) = 1$ for all u in $U_{n-1} \cap H_{p,q-1}$ if and only if $g_{n-2,j} = 0$ for $j = 0, 1, \ldots, p-q, p-q+1$ and $j = p-q+3, p-q+5, \ldots, p+q-5, p+q-3$. It is equivalent to say that g belongs to $P_{n-2}H_{p-1,q-1}$. $\square$

Proposition 3.1. *Let σ belong to Alg(P_{n-1}), and χ be a positive character of $P_n \cap H_{p,q}$, then there is a positive character χ' of $P_{n-1} \cap H_{p,q-1}$, such that*

$$Hom_{P_n \cap H_{p,q}}\big(\Phi^+\sigma, \chi\big) \hookrightarrow Hom_{P_{n-1}\cap H_{p,q-1}}\big(\sigma, \chi'\big).$$

Proof. Let V be the space on which σ acts, and $W = \phi^+ V$. Let A the projection from $\mathcal{C}_c^\infty(P_n, V)$ onto W, defined by $A(f(p)) = \int_{P_{n-1}U_n} \delta_{U_n}^{-1/2}(y)\sigma(y^{-1}) f(yg)\,\mathrm{d}y$. Lifting through A gives a vector space injection of $Hom_{P_n\cap H_{p,q}}(\Phi^+\sigma, \chi)$ into the space of V-distributions T on P_n satisfying relations

$$T \circ R(h_0) = \chi(h_0)T, \tag{1}$$

$$T \circ L(y_0) = \delta_{U_n}^{3/2}(y_0) T \circ \sigma(y_0) \tag{2}$$

for h_0 in $P_n \cap H_{p,q}$ and $y_0 \in P_{n-1}U_n$.

We introduce Θ the map on P_n defined by $\Theta(ug) = \theta(u)$ for u in U_n and g in G_{n-1}. Then the V-distribution $\Theta.T$ is U_n-invariant, hence there is a V-distribution S with support in G_{n-1} such that $\Theta.T = du \otimes S$ (where du denotes a Haar measure on U_n), and thus $T = \Theta^{-1}.du \otimes S$ has support $U_n.\mathrm{supp}(S)$. It is easily verified that $du \otimes S$ is right invariant under U_n, but because of relation (1), T is right invariant under $(U_n \cap H_{p,q})$. We deduce from these two facts that for g in supp(S), $\Theta(gu)$ must be equal to $\Theta(g)$ for any u in $U_n \cap H_{p,q}$. This means that supp$(S) \subset S_{p,q}$, and $S_{p,q} = P_{n-1}H_{p,q-1}$ according to Lemma 3.1, hence T has support in $P_{n-1}U_nH_{p,q-1}$.

Now consider the projection $B : \mathcal{C}_c^\infty(P_{n-1}U_n \times H_{p,q-1}, V) \twoheadrightarrow \mathcal{C}_c^\infty(P_{n-1}U_nH_{p,q-1}, V)$, defined by $B(\phi)(y^{-1}h) = \int_{P_{n-1}\cap H_{p,q-1}} \phi(ay, ah)\,\mathrm{d}a$ (which is well defined because of the equality $P_{n-1}U_n \cap H_{p,q-1} = P_{n-1} \cap H_{p,q-1}$), and $\phi \mapsto \tilde{\phi}$ the isomorphism of $\mathcal{C}_c^\infty(P_{n-1}U_n \times H_{p,q-1}, V)$ defined by $\tilde{\phi}(y,h) = \chi(h)\delta_{U_n}(y)^{3/2}\sigma(y)\phi(y,h)$.

If one sets $D(\phi) = T(B(\tilde{\phi}))$, then D is a V-distribution on $P_{n-1}U_n \times H_{p,q-1}$ which is right invariant under $P_{n-1}U_n \times H_{p,q-1}$. This implies that there exists a unique linear form λ on V, such that for all $D(\phi) = \int_{P_{n-1}U_n\times H_{p,q-1}} \lambda(\phi(y,h))\,\mathrm{d}y\,\mathrm{d}h$.

Now for b in $P_{n-1} \cap H_{p,q-1}$, on has from the integral expression of D, the relation $D \circ L(b,b) = \delta(b)D$ for some positive modulus character δ. On the other hand, writing D as $\phi \mapsto T(B(\tilde{\phi}))$, one has $\widetilde{L(b,b)\phi} = \chi(b)\delta_{U_n}^{3/2}(b)L(b,b)(\widetilde{\sigma(b^{-1})\phi})$ and $B \circ L(b,b) = \delta_1(b)B$ for a positive modulus character δ_1, so that $D \circ L(b,b) = \delta_1(b)\chi(b)\delta_{U_n}^{3/2}(b)D \circ \sigma(b^{-1})$. Comparing the two expressions for $D \circ L(b,b)$, we get the relation $D \circ \sigma(b) = \chi'(b)D$, with χ' being the positive character $\delta^{-1}\delta_1\chi\delta_{U_n}^{3/2}$ of $P_{n-1} \cap H_{p,q-1}$.

This in turn implies that the linear form λ on V satisfies the same relation, i.e. belongs to $Hom_{P_{n-1}\cap H_{p,q-1}}(\sigma, \chi')$, and $T \mapsto \lambda$ gives a linear injection of $Hom_{P_n\cap H_{p,q}}(\Phi^+\sigma, \chi)$ into $Hom_{P_{n-1}\cap H_{p,q-1}}(\sigma, \chi')$, and this proves the proposition. □

Using Lemma 3.2 instead of Lemma 3.1 in the previous proof, one obtains the following statement.

Proposition 3.2. *Let σ' belong to Alg(P_{n-2}), and χ' be a positive character of $P_{n-1} \cap H_{p,q-1}$, then there is a positive character χ'' of $P_{n-2} \cap H_{p-1,q-1}$, such that*

$$Hom_{P_{n-1}\cap H_{p,q-1}}\big(\Phi^+\sigma', \chi'\big) \hookrightarrow Hom_{P_{n-2}\cap H_{p-1,q-1}}\big(\sigma, \chi''\big).$$

A consequence of these two propositions is the following.

Proposition 3.3. *Let $n \geqslant 3$, and p and q two integers with $p+q=n$ and $p-1 \geqslant q \geqslant 0$, then one has $Hom_{P_n\cap H_{p,q}}((\Phi^+)^{n-1}\Psi^+(\mathbf{1}), \mathbf{1}_{P_n\cap H_{p,q}}) = \{0\}$.*

Proof. Using repeatedly the last two propositions, we get the existence of a positive character χ of P_{p-q+1} such that $Hom_{P_n\cap H_{p,q}}((\Phi^+)^{n-1}\Psi^+(\mathbf{1}), \mathbf{1}_{P_n\cap H_{p,q}}) \hookrightarrow Hom_{P_{p-q+1}\cap H_{p-q+1,0}}((\Phi^+)^{p-q}\Psi^+(\mathbf{1}), \chi) = Hom_{P_{p-q+1}}((\Phi^+)^{p-q}\Psi^+(\mathbf{1}), \chi)$, and this last space is 0 because $(\Phi^+)^{p-q}\Psi^+(\mathbf{1})$ and χ are two non-isomorphic irreducible representations of P_{p-q+1}, according to Corollary 3.5 of [1]. □

This implies the following theorem about cuspidal representations.

Theorem 3.1. *Let π be a cuspidal representation of G_n, which is distinguished by a maximal Levi subgroup M, then n is even and $M \simeq M_{n/2,n/2}$.*

Proof. Let M be the maximal Levi subgroup such that π is M-distinguished. Then M is conjugate to a standard Levi subgroup $M_{p,q}$ with $p \geqslant q$ and $p+q=n$. Suppose $p \geqslant q+1$, $M_{p,q}$ is conjugate to $H_{p,q}$, so that π is $H_{p,q}$-distinguished, and $\pi_{|P_n}$ is thus $H_{p,q} \cap P_n$-distinguished. But by Proposition 2.1, the restriction $\pi_{|P_n}$ is isomorphic to $(\Phi^+)^{n-1}\Psi^+(\mathbf{1})$, and this contradicts Proposition 3.3. Hence one must have $p = q$, and this proves the theorem. □

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