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On the compatibility between cup products, the Alekseev–Torossian connection and the Kashiwara–Vergne conjecture, II

Compatibilité entre cup-produits, connexion d'Alekseev–Torossian et conjecture de Kashiwara–Vergne, II

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ABSTRACT

We give a different proof of the famous result on compatibility between cup product (Kontsevich, 2003, [3, Section 8]) in cohomology of degree 0, for a finite-dimensional Lie algebra, from which we deduce an alternative way of re-writing Kontsevich's star product by means of the Alekseev–Torossian connection (Alekseev and Torossian, 2010, [1]).

RÉSUMÉ

On donne ici une preuve différente de la compatibilité entre cup produits (Kontsevich, 2003, [3, Section 8]) en cohomologie de degré 0 dans le cas d'une algèbre de Lie de dimension finie, d'où on déduit, en utilisant la connection de Alekseev–Torossian (Alekseev et Torossian, 2010, [1]), une écriture alternative du produit-étoilé de Kontsevich.

1. The 1-form governing the compatibility between cup products

We consider $\mathfrak{g}$ as in [4, Section 2] and the associated Poisson variety $(X=\mathfrak{g}^*,\pi)$. Since π is linear, we may safely set $\hbar=1$ and consider the associative algebra $(A,\star)$, for $\star$ as in [4, Formula (2)].

For a non-negative integer n, let us consider the projection $\pi_{n,2}$ from $C^+_{n+2,0}$ onto $C^+_{2,0}$ which forgets all points in $\mathbb{H}^+$ except the last two: it extends smoothly to a projection from $\overline{C}^+_{n+2,0}$ onto $\overline{C}^+_{2,0}$, which we denote by the same symbol. It is clear that $\pi_{n,2}$ defines a fibration onto $\overline{C}^+_{2,0}$, whose typical fiber is a smooth, oriented manifold with corners of dimension $2n$.

With Γ in $\mathcal{G}_{n+2,0}$ such that $|E(\Gamma)|=2n$, we associate a smooth 0-form on $C^+_{2,0}$ with values in the bidifferential operators on A *via*

$$\mathcal{T}^\pi_\Gamma(f_1,f_2)=\mu_{n+2}\big(\pi_{n,2,*}\big(\omega_{\tau,\Gamma}(\underbrace{\pi\otimes\cdots\otimes\pi}_{n}\otimes f_1\otimes f_2)\big)\big)=\widehat{\varpi}_\Gamma\big(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)(f_1,f_2),\quad \widehat{\varpi}_\Gamma=\pi_{n,2,*}(\omega_\Gamma),\tag{1}$$

where $\pi_{n,2,*}$ denotes the integration along the fiber of the operator-valued form $\omega_{\tau,\Gamma}$ w.r.t. the projection $\pi_{n,2}$. We finally set

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$$\mathcal{T}^\pi(f_1,f_2)=\sum_{n\geqslant 0}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\ |E(\Gamma)|=2n}}\mathcal{T}^\pi_\Gamma(f_1,f_2),\quad f_i\in A,\ i=1,2. \tag{2}$$

Formula (2) yields a well-defined smooth function on $C^+_{2,0}$ with values in the bidifferential operators on A.

Proof of Theorem 3.2, [4]. First of all, for Γ in $\mathcal{G}_{n+2,0}$ such that $|E(\Gamma)|=2n$, $n\geqslant 1$, let us compute

$$\mathrm{d}\big(\mathcal{T}^\pi_\Gamma(f_1,f_2)\big)=\mathrm{d}\widehat{\varpi}_\Gamma\big(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)(f_1,f_2)=\pi^\partial_{n,2,*}(\omega_\Gamma)\big(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)(f_1,f_2),$$

where the second equality follows by means of the generalized Stokes Theorem for integration along the fiber, and $\pi^\partial_{n,2,*}$ denotes integration along the boundary of the compactification of the typical fiber of the projection $\pi_{n,2}$.

The boundary strata of codimension 1 of the compactification of the typical fiber of $\pi_{n,2}$ can be deduced from the boundary strata of codimension 1 of $\overline{C}^+_{n+2,0}$:

i) there is a subset A of $[n+2]=\{1,\dots,n\}$, $1\leqslant|A|\leqslant n$ which contains either $n+1$ or $n+2$ or neither of them, such that points in $\mathbb{H}^+$ labeled by A collapse either to the $n+1$-st or $n+2$-nd or to a point in $\mathbb{H}^+$ different from $n+1$ and $n+2$;

ii) there is a subset A of $[n+2]$, which either contains both $n+1$, $n+2$ or contains neither of them, such that the points in $\mathbb{H}^+$ labeled by A approach $\mathbb{R}$.

For Γ as above, we denote by Γ_A the subgraph of Γ, whose edges have both endpoints labeled by A.

The boundary strata of type ii) yield trivial contributions. Namely, let us consider first a subset A such that $n+1$, $n+2\notin A$: Fubini's Theorem implies that

$$\pi^{\partial,A}_{n,2,*}(\omega_\Gamma)\propto\int_{\overline{C}^+_{A,0}}\omega_{\Gamma_A},$$

and the properties of ω imply that the form degree of ω_{Γ_A} equals $2|A|$, while the dimension of $C^+_{A,0}$ equals $2|A|-2$. If A contains both $n+1$, $n+2$, we may repeat the previous arguments *verbatim* by replacing A by A^c.

Let us consider a general boundary stratum of type i) such that A contains neither $n+1$ nor $n+2$: Fubini's Theorem and [3, Lemma 6.6] imply that the corresponding contribution vanishes, if $|A|\geqslant 3$. In the case $|A|=2$, the only non-trivial contribution corresponds to Γ_A with one single edge, whence the boundary property i) of ω in [4] and integration of vol_{S^1} over S^1 yields

$$\pi^{\partial,A}_{n,2,*}(\omega_\Gamma)=\pi_{n-1,2,*}(\omega_{\Gamma/\Gamma_A}).$$

Observe that Γ/Γ_A, obtained by shrinking Γ_A to a single vertex, belongs to $\mathcal{G}_{n+1,0}$, no edge departs from $n+1$, $n+2$ and all other vertices are bivalent except one, which is trivalent (here, the valence of a vertex is the number of outgoing edges from the said vertex).

Finally, let us consider a boundary stratum of type i), labeled by a subset $n+1\in A$, $n+2\notin A$. Again by means of [3, Lemma 6.6] and the boundary property i) of ω, the only non-trivial contribution comes from Γ_A consisting of a single edge with endpoint $n+1$ and initial point different from $n+2$, whence as before

$$\pi^{\partial,A}_{n,2,*}(\omega_\Gamma)=\pi_{n-1,2,*}(\omega_{\Gamma/\Gamma_A}).$$

Due modifications of the previous arguments yield a similar formula in the situation $n+1\notin A$, $n+2\in A$. Here, Γ/Γ_A, if $n+1$ is in A, belongs to $\mathcal{G}_{n+1,0}$, exactly one edge departs from $n+1$, no edge departs from $n+2$, and all other vertices are bivalent; when $n+2$ belongs to A, Γ/Γ_A is described in a similar way by switching $n+1$ and $n+2$.

The previous computations yield

$$\begin{aligned}\mathrm{d}\big(\mathcal{T}^\pi(f_1,f_2)\big)&=\sum_{n\geqslant 1}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\ |E(\Gamma)|=2n}}\sum_{\substack{A\subseteq[n+2],\,|A|=2\\ n+1\in A,\,n+2\notin A}}\widehat{\varpi}_{\Gamma/\Gamma_A}\big(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)(f_1,f_2)\\&+\sum_{n\geqslant 1}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\ |E(\Gamma)|=2n}}\sum_{\substack{A\subseteq[n+2],\,|A|=2\\ n+1\notin A,\,n+2\in A}}\widehat{\varpi}_{\Gamma/\Gamma_A}\big(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)(f_1,f_2)\\&+\sum_{n\geqslant 1}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\ |E(\Gamma)|=2n}}\sum_{\substack{A\subseteq[n+2],\,|A|=2\\ n+1,n+2\notin A}}\widehat{\varpi}_{\Gamma/\Gamma_A}\big(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)(f_1,f_2)\end{aligned}$$

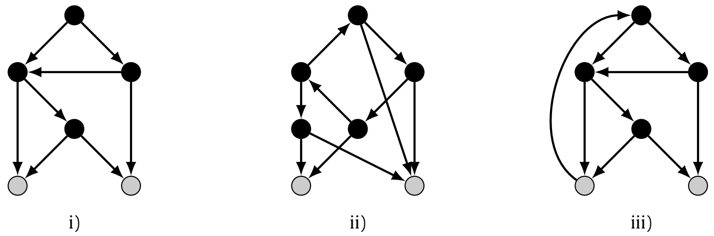

Fig. 1. i) A rooted, bivalent tree in $\mathcal{G}_{6,0}$, ii) a wheel-like graph with a bivalent rooted tree in $\mathcal{G}_{7,0}$, iii) a rooted, bivalent tree in $\mathcal{G}_{6,0}$ with an edge connecting the first external vertex to the root.

Fig. 1. i) Un arbre bivalent enraciné dans $\mathcal{G}_{6,0}$, ii) un graphe de type roue dans $\mathcal{G}_{7,0}$ avec un arbre bivalent enraciné y-attaché, iii) un arbre bivalent dans $\mathcal{G}_{6,0}$ avec une arête joignante le premier sommet exterieur à la racine.

$$\begin{aligned}
&= \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\ |E(\Gamma)|=2n+1}} \widehat{\varpi}_\Gamma\big(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)\big([\pi,f_1],f_2\big)\\
&\quad + \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\ |E(\Gamma)|=2n+1}} \widehat{\varpi}_\Gamma\big(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)\big(f_1,[\pi,f_2]\big)\\
&\quad + \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\ |E(\Gamma)|=2n+1}} \widehat{\varpi}_\Gamma\big(\mathcal{B}_\Gamma\big([\pi,\pi],\underbrace{\pi,\dots,\pi}_{n-1}\big)\big)(f_1,f_2),
\end{aligned} \tag{3}$$

recalling the explicit shape of the quotient subgraph Γ/Γ_A in the three previous cases and using Leibniz' rule to re-write the sums over A in the bidifferential operators; $[\pi,\pi]$ denotes the trivector field on x, whose components are given by the sum over the cyclic permutations of $\{j,k,l\}$ in $\pi_{ij}\partial_i\pi_{kl}$.

The third term in the final expression of (3) vanishes because of the Jacobi identity.

If we consider a general graph Γ in $\mathcal{G}_{n+2,0}$ as in the first term on the right-hand side of (3), any bivalent vertex different from $n+1$, $n+2$ may be the endpoint of at most one arrow because of the linearity of π. Thus, by slightly adapting the arguments of [2, Subsections 3.1.2–3.1.4], Γ factorizes uniquely into the union of its simple components,[1] which are depicted in Fig. 1 (the two gray-shaded vertices of the first type are called external, while the remaining vertices of the first type are called internal). Observe that the external vertices are only endpoints of edges, while the internal vertices have exactly two outgoing edges and one ingoing edge, except the root in i). By definition, Γ has exactly one simple component of type iii).

Let us consider a simple graph ${}_{\Rsh}\Gamma$, resp. $\Gamma_{\Lsh}$, of type iii) with exactly one edge connecting $n+1$, resp. $n+2$, to the root: then, borrowing previous notation, we may define

$$\Omega^\pi_{1,{}_{\Rsh}\Gamma}(f_1\otimes\xi,f_2) = \widehat{\varpi}_{{}_{\Rsh}\Gamma}\big((\xi\otimes 1\otimes 1)\circ(\mu_n\otimes 1\otimes 1)\circ\tau_\Gamma\big)(\underbrace{\pi\otimes\cdots\otimes\pi}_{n}\otimes f_1\otimes f_2), \tag{4}$$

$$\Omega^\pi_{2,\Gamma_{\Lsh}}(f_1,f_2\otimes\xi) = \widehat{\varpi}_{\Gamma_{\Lsh}}\big((\xi\otimes 1\otimes 1)\circ(\mu_n\otimes 1\otimes 1)\circ\tau_\Gamma\big)(\underbrace{\pi\otimes\cdots\otimes\pi}_{n}\otimes f_1\otimes f_2), \tag{5}$$

where f_i in A, $i=1,2$, ξ in $\mathfrak{g}^*$, and Γ is the rooted, bivalent tree obtained from ${}_{\Rsh}\Gamma$ or $\Gamma_{\Lsh}$ by removing the edge from $n+1$ or $n+2$ to the root.

Observe that $\widehat{\varpi}_{{}_{\Rsh}\Gamma}$ and $\widehat{\varpi}_{\Gamma_{\Lsh}}$ are well-defined, smooth 1-forms on $C^+_{2,0}$. Further, since Γ is a rooted, bivalent tree, $(\mu_n\otimes 1\otimes 1)\circ\tau_\Gamma$ is a linear map from $A^{\otimes 2}$ to $\mathfrak{g}\otimes A^{\otimes 2}$: hence, contraction of $\mathfrak{g}$ with $\mathfrak{g}^*$ yields an endomorphism of $A^{\otimes 2}$ consisting of differential operators with constant coefficients (and possibly infinite order). Summing up over all simple graphs of type iii) (4) and (5) we obtain well-defined, smooth 1-forms Ω^π_i, $i=1,2$, on $C^+_{2,0}$ with values in $\mathfrak{g}\otimes\widehat{\mathrm{S}}(\mathfrak{g}^*)^{\otimes 2}$, where we identify $\widehat{\mathrm{S}}(\mathfrak{g}^*)$ with the algebra of differential operators on A with constant coefficients.

On the other hand, the sum over all simple graphs of type i) and ii) yields the bidifferential operator $\mathcal{T}^\pi(\bullet,\bullet)$ by the arguments of [2, Subsubsections 3.1.2–3.1.4].

Therefore, Fubini's Theorem and the decomposition of admissible graphs into simple components of type i), ii) and iii) yield

$$\mathrm{d}\big(\mathcal{T}^\pi(f_1,f_2)\big) = \mathcal{T}^\pi\big(\Omega^\pi_1\big([\pi,f_1],f_2\big)\big) + \mathcal{T}^\pi\big(\Omega^\pi_2\big(f_1,[\pi,f_2]\big)\big), \quad f_i\in A,\ i=1,2. \tag{6}$$

The 0-form $\mathcal{T}^\pi$ and the 1-forms Ω^π_i, $i=1,2$, are smooth on $C^+_{2,0}$ and extend to the class L^1 when restricted on piecewise differentiable curves on $\overline{C}^+_{2,0}$.

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[1] An element Γ of $\mathcal{G}_{n,2}$ is simple, if the graph obtained from Γ by removing all arrows connecting to the vertices of the second type is connected. In the present situation, we may regard Γ in $\mathcal{G}_{n+2,0}$ as an element of $\mathcal{G}_{n,2}$ by interpreting the last two vertices of the first type as vertices of the second type.

Now, let us evaluate $\mathcal{T}^\pi(f_1, f_2)$ at a point in the boundary stratum $\bar{C}_2 = S^1$ of $\bar{C}^+_{2,0}$ (*i.e.* the two distinct points in $\mathbb{H}^+$ collapse together along a prescribed direction). The skew-symmetry of π and the result of [5] imply that the only non-trivial contribution comes from the unique graph in $\mathcal{G}_{2,0}$ with no edges.

Let us evaluate $\mathcal{T}^\pi(f_1, f_2)$ at the boundary stratum $\bar{C}^+_{0,2} = \{0, 1\}$ of codimension 2 of $\bar{C}^+_{2,0}$ (*i.e.* the two distinct points in $\mathbb{H}^+$ approach 0 and 1 on $\mathbb{R}$). Dimensional reasons, the linearity of π and the main result of [5] imply that $\mathcal{T}^\pi(f_1, f_2)$ evaluated at $\bar{C}^+_{0,2} = \{0, 1\}$ equals $f_1 \star f_2$.

If we now consider a piecewise differentiable curve γ on $\bar{C}^+_{2,0}$ connecting the chosen point in $\bar{C}_2 = S^1$ with $\bar{C}^+_{0,2} = \{0, 1\}$, and whose interior is in $C^+_{2,0}$, integration of (6) along γ yields

$$f_1 \star f_2 - f_1 f_2 = \int_\gamma \left(\mathcal{T}^\pi\left(\Omega^\pi_1\left([\pi, f_1], f_2\right)\right) + \mathcal{T}^\pi\left(\Omega^\pi_2\left(f_1, [\pi, f_2]\right)\right)\right), \tag{7}$$

hence the proof of [4, Theorem 3.2], whose claim is precisely a special case of the famous compatibility between cup products [3, Theorem 8.2]. □

2. Relationship with the AT connection

By their very construction, $\mathcal{T}^\pi$ and Ω^π_i, $i = 1, 2$, extend to the completed symmetric algebra $\widehat{A} = \widehat{\mathrm{S}}(\mathfrak{g}) = \mathbb{K}[[x_1, \ldots, x_d]]$. For y_i, $i = 1, 2$, in $\mathfrak{g}$, we consider e^{y_i} in $\widehat{A}$.

A direct computation recalling Formulæ (4), (5) yields

$$\Omega^\pi_1\left(e^{y_1} \otimes \xi, e^{y_2}\right) = \langle \xi, \omega_1(y_1, y_2)\rangle e^{y_1} \otimes e^{y_2}, \qquad \Omega^\pi_2\left(e^{y_1} \otimes \xi, e^{y_2}\right) = \langle \xi, \omega_2(y_1, y_2)\rangle e^{y_1} \otimes e^{y_2},$$

where ω_i denotes here the AT connection on $C^+_{2,0}$ [6,1].

Following the same patterns, it is not difficult to prove by direct computations the following identities:

$$\Omega^\pi_1\left(\left[\pi, e^{y_1}\right], e^{y_2}\right) = \left\langle \left[y_1, \omega_1(y_1, y_2)\right], \partial_{y_1}\right\rangle\left(e^{y_1}\right) \otimes e^{y_2} + \mathrm{tr}_\mathfrak{g}\left(\mathrm{ad}(y_1)\partial_{y_1}\omega_1(y_1, y_2)\right)e^{y_1} \otimes e^{y_2},$$
$$\Omega^\pi_2\left(\left[\pi, e^{y_1}\right], e^{y_2}\right) = e^{y_1} \otimes \left\langle \left[y_2, \omega_2(y_1, y_2)\right], \partial_{y_2}\right\rangle\left(e^{y_2}\right) + \mathrm{tr}_\mathfrak{g}\left(\mathrm{ad}(y_2)\partial_{y_2}\omega_2(y_1, y_2)\right)e^{y_1} \otimes e^{y_2},$$

where $\mathrm{tr}_\mathfrak{g}(\bullet)$ denotes the trace of endomorphisms of $\mathfrak{g}$, $\mathrm{ad}(\bullet)$ the adjoint representation of $\mathfrak{g}$ and $\partial_{y_1}\omega_1(y_1, y_2)$ the endomorphism of $\mathfrak{g}$ defined *via*

$$\left(\partial_{y_1}\omega_1(y_1, y_2)\right)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\omega_1(y_1 + tx, y_2)\bigg|_{t=0}, \quad x \in \mathfrak{g}.$$

It is possible to re-write (7) as

$$\begin{aligned} e^{y_1} \star e^{y_2} - e^{y_1}e^{y_2} &= \int_\gamma \left(\mathcal{T}^\pi\left(\left\langle\left[y_1, \omega_1(y_1, y_2)\right], \partial_{y_1}\right\rangle\left(e^{y_1}\right), e^{y_2}\right) + \mathcal{T}^\pi\left(e^{y_1}, \left\langle\left[y_1, \omega_1(y_1, y_2)\right], \partial_{y_1}\right\rangle\left(e^{y_2}\right)\right)\right) \\ &\quad + \left(\mathrm{tr}_\mathfrak{g}\left(\mathrm{ad}(y_1)\partial_{y_1}\omega_1(y_1, y_2)\right) + \mathrm{tr}_\mathfrak{g}\left(\mathrm{ad}(y_2)\partial_{y_2}\omega_2(y_1, y_2)\right)\right)\int_\gamma \mathcal{T}^\pi\left(e^{y_1}, e^{y_2}\right) \\ &= \int_\gamma \left(\left\langle\left[y_1, \omega_1(y_1, y_2)\right], \partial_{y_1}\right\rangle + \left\langle\left[y_2, \omega_2(y_1, y_2)\right], \partial_{y_1}\right\rangle + \mathrm{div}\left(\omega(y_1, y_2)\right)\right)D_\mathrm{T}(y_1, y_2)e^{Z_T(y_1, y_2)}, \end{aligned}$$

borrowing notation from [4, Section 4] and where, following notation from [1],

$$\mathrm{div}\left(\omega(y_1, y_2)\right) = \mathrm{tr}_\mathfrak{g}\left(\mathrm{ad}(y_1)\partial_{y_1}\omega_1(y_1, y_2)\right) + \mathrm{tr}_\mathfrak{g}\left(\mathrm{ad}(y_2)\partial_{y_2}\omega_1(y_1, y_2)\right).$$

Finally, by $D_\mathrm{T}(\bullet,\bullet)$ and $Z_\mathrm{T}(\bullet,\bullet)$ we denote the functions over $C^+_{2,0}$, providing deformations of the Duflo density function $D(\bullet,\bullet)$ and the Baker–Campbell–Hausdorff (shortly, BCH) formula $Z(\bullet,\bullet)$ respectively, introduced in [6].

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