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Algebraic Geometry

Rational curves on Fermat hypersurfaces

Courbes rationnelles sur des hypersurfaces de Fermat

Mingmin Shen

DPMMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK

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ABSTRACT

In this note we study rational curves on degree p^r+1 Fermat hypersurface in $\mathbb{P}_k^{p^r+1}$, where k is an algebraically closed field of characteristic p. The key point is that the presence of Frobenius morphism makes the behavior of rational curves to be very different from that of characteristic 0. We show that if there exists N_0 such that for all $e \geqslant N_0$ there is a degree e very free rational curve on X, then $N_0 > p^r(p^r-1)$.

RÉSUMÉ

Note nous étudions les courbes rationnelles sur les hypersurfaces de Fermat de degré p^r+1 dans $\mathbb{P}_k^{p^r+1}$, où k est un corps algébriquement clos de caractéristique p. Le point essentiel est la présence du morphisme de Frobenius qui rend le comportement des courbes rationnelles très différent du cas de caractéristique 0. Nous montrons que si N_0 est un entier tel que pour tout $e \geqslant N_0$ il y ait une courbe rationnelle très libre de degré e sur l'hypersurface de Fermat, alors $N_0 > p^r(p^r-1)$.

1. Introduction

Rational curves appear to be very important in the study of higher dimensional algebraic varieties. We refer to [4] for the background. Let X be a smooth projective variety over an algebraically closed field k.

Definition 1.1. A rational curve $f : C \cong \mathbb{P}^1 \to X$ is free (resp. very free) if f^*T_X is globally generated (resp. ample). We say that X is *separably rationally connected* (SRC) if there is a very free rational curve on X.

Definition 1.2. X is *rationally connected* (RC) if a general pair of points can be connected by a rational curve. This means that there is a family of rational curves $\pi : U \to Y$ together with a morphism $u : U \to X$ such that the natural map $u^{(2)} : U \times_Y U \to X \times_k X$ is dominant. If we only require the general fiber of π to be a genus 0 curve, then we say that X is rationally chain connected.

One very important tool to study rational curves is deformation theory. This works especially well in characteristic 0. For example, it is easy to see that SRC implies RC in any characteristic. But if the characteristic is 0, then RC is equivalent to SRC. One very important class of rationally connected varieties is provided by the following:

E-mail address: M.Shen@dpmms.cam.ac.uk.

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Theorem 1.3. *(See [5,2].) Smooth Fano varieties over a field of characteristic* 0 *are rationally connected.*

The case of characteristic p is still mystery. We know that all smooth Fano varieties are rationally chain connected, see V.2 of [4]. Kollár has constructed examples of singular Fano's that are not SRC, see V.5 of [4]. This naturally leads to the question whether all smooth Fano varieties are SRC. Recently Y. Zhu has proved that a general Fano hypersurface is SRC, see [6].

In this note, we consider a class of very special Fano hypersurfaces over a field of positive characteristic. From now on, we fix k to be an algebraically closed field of characteristic $p > 0$. Let $X = X_{d,N} \subset \mathbb{P}^N_k$ be the Fermat hypersurface defined by

$$X_0^d + X_1^d + \cdots + X_N^d = 0$$

where $d = p^r + 1 > 3$ and $N \geqslant 3$. These hypersurfaces are special since they are always unirational, see Théorèm 3.8 of [3]. Note that for fixed N, the hypersurface $X_{d,N}$ is of general type if $d > N$. Hence they are examples of varieties of general type which are unirational. The variety X is Fano if and only if $d \leqslant N$. This condition is necessary for X to have a very free rational curve. In this case, we ask the following:

Question 1.4. If $d \leqslant N$, is there a very free rational curve on X?

A positive answer for the cases when $N \geqslant 2p^r - 1$ was given by Corollaire 3.17 of [3]. The following lemma shows that a positive answer for large N can always be deduced from a positive answer for the case of a smaller N:

Lemma 1.5. *If $X_{d,N}$ contains a very free rational curve for $N = p^r + 1$, then $X_{d,N}$ contains a very free rational curve for all $N \geqslant p^r + 1$.*

Hence we see that the most interesting case is when $d = N$. Our first observation is

Proposition 1.6. *Let $X = X_{d,d}$ be the Fermat hypersurface of degree $d = p^r + 1$ in $\mathbb{P}^d_k$ and M_e be the space of degree e morphisms from $\mathbb{P}^1$ to X. Then for M_e to have the expected dimension, e has to be at least $p^r - 1$. In particular, if $e < p^r - 1$ then there is no free rational curve of degree e.*

This is a special case of Théorème 3.16(a) of [3]. For very free rational curves on $X_{d,d}$ we have the following:

Theorem 1.7. *Let $X = X_{d,d}$ be the Fermat hypersurface of degree $d = p^r + 1$ in $\mathbb{P}^d_k$. Let $f : C = \mathbb{P}^1 \to X$ be a rational curve of degree e. If $mN < e \leqslant (m + 1)(N - 1)$ for some $0 \leqslant m \leqslant N - 3$, then f is not very free.*

The case when $m = 0$ can be deduced from Théorème 3.16(a) of [3].

Corollary 1.8. *If there exists N_0 such that for all $e \geqslant N_0$ there is a very free rational curve of degree e on X, then $N_0 > p^r(p^r - 1)$.*

One of the simplest cases is when X is the degree 5 Fermat hypersurface in $\mathbb{P}^5$ over $\overline{\mathbb{F}}_2$. The above theorem shows that on this X very free rational curves can only exist in degrees 5, 9, 10 and all $e \geqslant 13$. In the recent paper [1], the authors showed that there is no very free rational curve in degree 5 and they explicitly gave a very free rational curve of degree 9.

Definition 1.9. A *rational normal curve of degree e* is a rational curve $f : \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ of degree e whose linear span is a subspace $\mathbb{P}^e \subset \mathbb{P}^N$.

In particular, a necessary (and also sufficient) condition to have a rational normal curve of degree e in $\mathbb{P}^N$ is that $e \leqslant N$. One candidate of a very free rational curve of low degree on X is given by the following:

Proposition 1.10. *Let $X = X_{d,N}$ be as above. If $C \subset X$ is a rational normal curve of degree N (viewed as a rational curve on $\mathbb{P}^N$), then C is very free on X.*

This was first obtained in Théorème 3.16(b) of [3]. Proposition 3.19 of [3] shows that such rational normal curves exist only for $N \geqslant 2p^r - 1$.

2. Proofs

Let $X = X_{d,N}$. We use the following diagram to investigate the tangent sheaf of X:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & T_X & \longrightarrow & T_{\mathbb{P}^N}|_X & \longrightarrow & \mathcal{O}_X(p^r+1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathcal{O}_X(1)^{N+1} & \xrightarrow{(X_i^{p^r})} & \mathcal{O}_X(p^r+1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle (X_i)} & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{1}
$$

We dualize the second column and get

$$0 \longrightarrow \Omega^1_{\mathbb{P}^N} \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X^{N+1} \xrightarrow{(X_i)} \mathcal{O}_X(1) \longrightarrow 0$$

Let $F : X \to X$ be the Frobenius morphism. We apply $(F^*)^r$ to the above exact sequence and get

$$0 \longrightarrow (F^*)^r \Omega^1_{\mathbb{P}^N} \otimes \mathcal{O}_X(p^r) \longrightarrow \mathcal{O}_X^{N+1} \xrightarrow{(X_i^{p^r})} \mathcal{O}_X(p^r) \longrightarrow 0 \tag{2}$$

Compare this with the second row of (1), we get

$$\mathscr{F} \cong \left(F^*\right)^r \Omega^1_{\mathbb{P}^N} \otimes \mathcal{O}_X\left(p^r+1\right) \tag{3}$$

Proof of Proposition 1.10. Since $f : C \to \mathbb{P}^N$ is a rational normal curve, we have

$$f^*\Omega^1_{\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^1}(-N-1)^{\oplus N}$$

Hence

$$f^*\mathscr{F} = \mathcal{O}_{\mathbb{P}^1}\left((-N-1)p^r + \left(p^r+1\right)N\right)^{\oplus N} = \mathcal{O}_{\mathbb{P}^1}\left(N-p^r\right)^N$$

is very ample. The first column of diagram (1) shows that f^*T_X is very ample. □

Proof of Theorem 1.7. Consider the first column in diagram (1), we know that if the splitting of $f^*\mathscr{F}$ has a negative summand or has at least two copies of $\mathcal{O}_{\mathbb{P}^1}$, then f^*T_X is not ample. To avoid this from happening, the best situation is when $\Omega^1_{\mathbb{P}^N}|_C$ is balanced. Now we assume that all the above splittings are balanced. Let $a = [\frac{(N+1)e}{N}] = e + [\frac{e}{N}]$. Then

$$\Omega^1_{\mathbb{P}^N}\big|_C \cong \mathcal{O}(-a)^l \oplus \mathcal{O}(-a-1)^{l'}$$

with $l' = (N+1)e - Na$ and $l = N - l'$. Then it follows from (3) that

$$f^*\mathscr{F} \cong \mathcal{O}(b_1)^l \oplus \mathcal{O}(b_2)^{l'}$$

with $b_1 = -ap^r + e(p^r+1)$ and $b_2 = -(a+1)p^r + e(p^r+1)$. Note that $f^*\mathscr{F}$ is highly unbalanced unless e is a multiple of N. This is because when e is a multiple of N, then $l' = 0$ and there is no summand of the form $\mathcal{O}(b_2)$ in the above splitting. If $mN < e < (m+1)N$, then $a = e + m$ and

$$b_2 = -(a+1)p^r + e\left(p^r+1\right) = e - (m+1)(N-1)$$

Hence we have $b_2 < 0$ if $e < (m+1)(N-1)$. If $e = (m+1)(N-1)$, then

$$l' = (N+1)e - Na = N - m - 1$$

The theorem follows easily from this computation. □

Proof of Proposition 1.6. A degree e rational curve $f : \mathbb{P}^1 \to X$ can be written as

$$t \mapsto [f_0 : f_1 : \cdots : f_d]$$

where $f_i = \sum_{j=0}^{e} a_{ij} t^j$ are polynomials of degree at most e. The condition for the image of f to be contained in X is given by $F = \sum f_i^d = 0$ as a polynomial in t. If f is free, then Riemann–Roch tells us that $h^0(\mathbb{P}^1, f^* T_X) = d + e - 1$ which is, by definition, the expected dimension of M_e. By explicit computation we have

$$f_i^d = \left(\sum_{j=0}^{e} a_{ij} t^j\right)^{p^r} \left(\sum_{j=0}^{e} a_{ij} t^j\right) = \left(\sum_{j=0}^{e} a_{ij}^{p^r} t^{jp^r}\right)\left(\sum_{j=0}^{e} a_{ij} t^j\right)$$

If $e < p^r - 1$, then the coefficient of $t^{jp^r - 1}$, $j = 1, \ldots, e$, is automatically 0. Hence the actual dimension of M_e is bigger than the expected dimension. □

Proof of Lemma 1.5. It is easy to realize $X_{d,N}$ as hyperplane section of $X_{d,N+1}$. This implies that any very free rational curve on $X_{d,N}$ gives a very free rational curve on $X_{d,N+1}$. □

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