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# Asymptotically exact Korn's constant for thin cylindrical domains

## *Développement asymptotique précis de la constante de Korn dans une poutre mince*

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A B S T R A C T

We consider a cylinder $\Omega^\varepsilon$ having fixed length and small cross-section $\varepsilon\omega$ with $\omega \subset \mathbb{R}^2$. Let $1/K^\varepsilon$ be the Korn constant of $\Omega^\varepsilon$. We show that, as $\varepsilon$ tends to zero, $K^\varepsilon/\varepsilon^2$ converges to a positive constant. We provide a characterization of this constant in terms of certain parameters that depend on $\omega$.



R É S U M É

On considère une poutre verticale $\Omega^\varepsilon$ de hauteur fixée et de petite section $\varepsilon\omega$ avec $\omega \subset \mathbb{R}^2$. Soit $1/K^\varepsilon$ la constante de Korn dans $\Omega^\varepsilon$. On démontre que, lorsque $\varepsilon$ tend vers zéro, $K^\varepsilon/\varepsilon^2$ converge vers une constante positive. On caractérise la limite en fonction de paramètres qui dépendent de $\omega$.



## 1. Introduction and results

Given a domain $\Omega \subset \mathbb{R}^3$, Korn's inequality [13]:

$$\int_\Omega |\nabla \boldsymbol{u}|^2 \,\mathrm{d}^3\boldsymbol{x} \leqslant C_K \int_\Omega \left|\boldsymbol{E}(\boldsymbol{u})\right|^2 \mathrm{d}^3\boldsymbol{x}, \quad \forall \boldsymbol{u} \in \mathcal{A} \subset H^1\left(\Omega;\mathbb{R}^3\right)$$

is the key estimate to establish the solvability of the boundary-value problem of linear elastostatics [2]. This estimate holds under fairly general assumptions on $\Omega$, provided that certain side conditions are imposed on the *displacement* $\boldsymbol{u}$ through the choice of the *admissible space* $\mathcal{A}$ (two examples are given in (2) below). It asserts that the $L^2$ norm of the *strain* $\boldsymbol{E}(\boldsymbol{u}) := \operatorname{sym}\nabla\boldsymbol{u}$ controls the $L^2$ norm of the *whole displacement gradient*. The optimal choice for *Korn's constant* $C_K$ is given by $1/K(\Omega,\mathcal{A})$, where

$$K(\Omega,\mathcal{A}) := \inf_{\boldsymbol{u}\in\mathcal{A}\setminus\{\boldsymbol{0}\}} \frac{\int_\Omega |\boldsymbol{E}(\boldsymbol{u})|^2\,\mathrm{d}^3\boldsymbol{x}}{\int_\Omega |\nabla\boldsymbol{u}|^2\,\mathrm{d}^3\boldsymbol{x}}.$$

A vast body of literature investigates the dependence of Korn's constant on the geometric properties of the domain. Estimates for thin domains, such as rods and plates, were obtained in [12,15,1,3,18,5,16,11]. Let us consider a family of *rod-like domains*:

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$$\Omega^\varepsilon = \varepsilon\omega \times (0,\ell) := \{\boldsymbol{x}^\varepsilon = (\varepsilon x_1, \varepsilon x_2, x_3) \in \mathbb{R}^3\colon\ (x_1,x_2) \in \omega,\ x_3 \in (0,\ell)\} \quad \text{with } \varepsilon > 0,$$

and let us set

$$\kappa^\varepsilon_\sharp(\omega,\ell) := \frac{1}{\varepsilon^2} K\big(\Omega^\varepsilon, \mathcal{A}^\varepsilon_\sharp\big) = \inf_{\boldsymbol{u}\in\mathcal{A}^\varepsilon_\sharp\setminus\{\boldsymbol{0}\}} \frac{\int_{\Omega^\varepsilon} |\boldsymbol{E}(\boldsymbol{u})|^2\,\mathrm{d}^3\boldsymbol{x}^\varepsilon}{\int_{\Omega^\varepsilon} |\varepsilon\nabla\boldsymbol{u}|^2\,\mathrm{d}^3\boldsymbol{x}^\varepsilon}, \tag{1}$$

where the subscript "$\sharp$" stands for either "$dd$" or "$dn$", with

$$\mathcal{A}^\varepsilon_{dd} = \{\boldsymbol{u} \in H^1(\Omega^\varepsilon;\mathbb{R}^3)\colon\ \boldsymbol{u}|_{x_3=0} = \boldsymbol{u}|_{x_3=\ell} = \boldsymbol{0}\}, \qquad \mathcal{A}^\varepsilon_{dn} = \{\boldsymbol{u} \in H^1(\Omega^\varepsilon;\mathbb{R}^3)\colon\ \boldsymbol{u}|_{x_3=0} = \boldsymbol{0}\}. \tag{2}$$

In this Note we show that

$$\lim_{\varepsilon\to 0} \kappa^\varepsilon_\sharp(\omega,\ell) = \kappa_\sharp(\omega,\ell), \quad \text{where } \kappa_{dd}(\omega,\ell) = \frac{\pi^2}{4\ell^2}\frac{J_t(\omega)}{A(\omega)} \text{ and } \kappa_{dn}(\omega,\ell) = \frac{\pi^2}{8\ell^2}\frac{J(\omega)}{A(\omega)}, \tag{3}$$

with

$$J_t(\omega) := \min_{\psi\in H^1(\omega)} \int_\omega (D_1\psi - x_2)^2 + (D_2\psi + x_1)^2\,\mathrm{d}x_1\,\mathrm{d}x_2, \qquad J(\omega) := \min\left\{J_1(\omega),\ J_2(\omega),\ \frac{J_t(\omega)}{2}\right\},$$

$$J_1(\omega) := \int_\omega x_2^2\,\mathrm{d}x_1\,\mathrm{d}x_2, \qquad J_2(\omega) := \int_\omega x_1^2\,\mathrm{d}x_1\,\mathrm{d}x_2, \qquad A(\omega) := \int_\omega 1\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

We point out that, while the limit $\kappa_{dd}$ depends on the cross-section simply through the ratio $J_t/A$, the dependence of $\kappa_{dn}$ on $\omega$ is more involved. For example, $J_t/2 = J_1$ for a circle, $J < J_t/2$ for an ellipsis, and $J_t/2 < J$ for a square. A detailed discussion of these examples can be found in [20].

## 2. Rescaling and $\Gamma$-convergence of Rayleigh's quotient

Our proof of (3) is based on $\Gamma$-convergence. Following the standard approach [4], we perform a change of variables. To this end, we set $\Omega = \Omega^1$, and $\mathcal{A}_\sharp = \mathcal{A}^1_\sharp$. Then, to every $\boldsymbol{u} \in \mathcal{A}^\varepsilon_\sharp$ we associate $\boldsymbol{v} \in \mathcal{A}_\sharp$ defined by $v_\alpha(\boldsymbol{x}) = \varepsilon u_\alpha(\boldsymbol{x}^\varepsilon)$ and $v_3(\boldsymbol{x}) = u_3(\boldsymbol{x}^\varepsilon)$, where $\boldsymbol{x} = (x_1,x_2,x_3) \in \Omega$ and $\boldsymbol{x}^\varepsilon = (\varepsilon x_1, \varepsilon x_2, x_3) \in \Omega^\varepsilon$. As a result, we can rewrite (1) as

$$\kappa^\varepsilon_\sharp(\omega,\ell) = \inf_{\boldsymbol{v}\in\mathcal{A}_\sharp\setminus\{\boldsymbol{0}\}} \mathcal{R}^\varepsilon(\boldsymbol{v}), \quad \text{where } \mathcal{R}^\varepsilon(\boldsymbol{v}) := \frac{\int_\Omega |\boldsymbol{E}^\varepsilon(\boldsymbol{v})|^2\,\mathrm{d}^3\boldsymbol{x}}{\int_\Omega |\varepsilon\nabla^\varepsilon\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{x}}, \quad \text{with}$$

$$\big(\nabla^\varepsilon\boldsymbol{v}\big)_{\alpha\beta} = \frac{v_{\alpha,\beta}}{\varepsilon^2}, \qquad \big(\nabla^\varepsilon\boldsymbol{v}\big)_{\alpha 3} = \frac{v_{\alpha,3}}{\varepsilon}, \qquad \big(\nabla^\varepsilon\boldsymbol{v}\big)_{3\alpha} = \frac{v_{3,\alpha}}{\varepsilon}, \qquad \big(\nabla^\varepsilon\boldsymbol{v}\big)_{33} = v_{3,3}, \qquad \boldsymbol{E}^\varepsilon(\boldsymbol{v}) = \operatorname{sym}\nabla^\varepsilon\boldsymbol{v},$$

where Greek indices run over $\{1,2\}$, and a comma denotes partial differentiation. We next introduce the spaces:

$$\mathcal{A}^{BN} := \{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)\colon\ E_{\alpha 3}(\boldsymbol{v}) = \boldsymbol{0}\},$$

$$H^1_{dn}(0,\ell) := \{f \in H^1(0,\ell)\colon\ f(0) = 0\}, \quad \text{and} \quad H^1_{dd}(0,\ell) := \{f \in H^1_{dn}(0,\ell)\colon\ f(\ell) = 0\},$$

and we prove:

**Theorem 2.1.** *Let the functional $\mathcal{R} : \mathcal{A}_\sharp \times H^1_\sharp(0,\ell) \to \mathbb{R}\cup\{+\infty\}$ be defined by*

$$\mathcal{R}(\boldsymbol{v},\theta) := \frac{\int_\Omega v_{3,3}^2 + \frac{J_t}{2A}(\theta')^2\,\mathrm{d}^3\boldsymbol{x}}{2\int_\Omega W_{13}^2(\boldsymbol{v}) + W_{23}^2(\boldsymbol{v}) + \theta^2\,\mathrm{d}^3\boldsymbol{x}} \quad \text{if } (\boldsymbol{v},\theta) \neq (\boldsymbol{0},0) \text{ and } \boldsymbol{v} \in \mathcal{A}_\sharp \cap \mathcal{A}^{BN} =: \mathcal{A}^{BN}_\sharp,$$

*and $\mathcal{R}^\varepsilon(\boldsymbol{v},\theta) := +\infty$ otherwise. The sequence $\mathcal{R}^\varepsilon$ $\Gamma$-converges to $\mathcal{R}$ in the following sense:*

(i) *for every sequence $\{\boldsymbol{v}^\varepsilon\} \subset \mathcal{A}_\sharp$ and for every $(\boldsymbol{v},\theta) \in \mathcal{A}_\sharp \times H^1_\sharp(0,\ell)$ such that $\boldsymbol{v}^\varepsilon \overset{H^1}{\rightharpoonup} \boldsymbol{v}$ and $(\varepsilon\nabla\boldsymbol{v}^\varepsilon)_{21} \overset{L^2}{\to} \theta$ we have that $\mathcal{R}(\boldsymbol{v},\theta) \leqslant \liminf_\varepsilon \mathcal{R}^\varepsilon(\boldsymbol{v}^\varepsilon)$;*

(ii) *for every $(\boldsymbol{v},\theta) \in \mathcal{A}_\sharp \times H^1_\sharp(0,\ell)$ there exists a sequence $\{\boldsymbol{v}^\varepsilon\} \subset \mathcal{A}_\sharp$ such that $\boldsymbol{v}^\varepsilon \overset{H^1}{\rightharpoonup} \boldsymbol{v}$, $(\varepsilon\nabla\boldsymbol{v}^\varepsilon)_{21} \overset{L^2}{\to} \theta$, and $\limsup_\varepsilon \mathcal{R}^\varepsilon(\boldsymbol{v}^\varepsilon) \leqslant \mathcal{R}(\boldsymbol{v},\theta)$.*

In order to prove the *liminf inequality* (i), we use the lower semicontinuity of the numerator of $\mathcal{R}^\varepsilon$ with respect to the weak convergence in $L^2$ of $\boldsymbol{E}^\varepsilon(\boldsymbol{v}^\varepsilon)$, and certain arguments of common use to derive rod theories (see for instance [1,14,24]). We also use the strong convergence of $\varepsilon\nabla^\varepsilon\boldsymbol{v}^\varepsilon$ in the denominator. To this aim we use the next theorem, where $\mathbb{R}^{3\times 3}_{\mathrm{skw}}$ is the space of skew-symmetric $3\times 3$ matrices and $\boldsymbol{W}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} - \nabla\boldsymbol{u}^T)$.

**Theorem 2.2.** *Let $\{\boldsymbol{v}^\varepsilon\} \subset \mathcal{A}_\sharp$ be such that $\sup_\varepsilon \|\boldsymbol{E}^\varepsilon(\boldsymbol{v}^\varepsilon)\|_{L^2} < +\infty$. Then, up to a subsequence, we have*

$$\boldsymbol{v}^\varepsilon \overset{H^1}{\rightharpoonup} \boldsymbol{v} \in \mathcal{A}_\sharp, \qquad \boldsymbol{E}^\varepsilon(\boldsymbol{v}) \overset{L^2}{\rightharpoonup} \boldsymbol{E}, \quad \text{with } E_{3i}(\boldsymbol{v}) = 0 \text{ and } E_{33}(\boldsymbol{v}) = E_{33}, \tag{4}$$

$$\varepsilon\nabla^\varepsilon \boldsymbol{v}^\varepsilon \overset{L^2}{\to} \boldsymbol{W} \in H^1\big(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{skw}}\big) \quad \text{with } W_{\alpha 3} = W_{\alpha 3}(\boldsymbol{v}). \tag{5}$$

*Moreover, there exist $\theta \in H^1_\sharp(0,\ell)$ and $\varphi \in L^2(0,\ell; H^1(\omega))$ such that*

$$W_{21}(\boldsymbol{x}) = \theta(x_3), \qquad 2E_{13}(\boldsymbol{x}) = \varphi_{,1}(x_1,x_2) - x_2\theta'(x_3), \qquad 2E_{23}(\boldsymbol{x}) = \varphi_{,2}(x_1,x_2) + x_1\theta'(x_3). \tag{6}$$

By a standard result from $\Gamma$-convergence, see [6], Theorem 2.1 and Theorem 2.2 imply that

$$\lim_{\varepsilon\to 0} \inf_{\boldsymbol{v}\in\mathcal{A}_\sharp\setminus\{\boldsymbol{0}\}} \mathcal{R}^\varepsilon(\boldsymbol{v}) = \min_{(\boldsymbol{v},\theta)\in\mathcal{A}^{BN}_\sharp\times H^1_\sharp(0,\ell)} \mathcal{R}(\boldsymbol{v},\theta).$$

It is shown in [14] that the *Bernoulli–Navier space* $\mathcal{A}^{BN}$ defined in the statement of Theorem 1 can be characterized as follows

$$\mathcal{A}^{BN} = \big\{\boldsymbol{v} \in H^1\big(\Omega;\mathbb{R}^3\big)\colon\ v_\alpha(\boldsymbol{x}) = w_\alpha(x_3),\ v_3(\boldsymbol{x}) = w_3(x_3) - x_\alpha w'_\alpha(x_3),\ w_\alpha \in H^2(0,\ell),\ w_3 \in H^1(0,\ell)\big\},$$

where a prime denotes differentiation. From this characterization we derive

$$\min_{(\boldsymbol{v},\theta)\in\mathcal{A}^{BN}_\sharp\times H^1_\sharp(0,\ell)} \mathcal{R}(\boldsymbol{v},\theta) = \min_{(\boldsymbol{w},\theta)\in A_\sharp\setminus\{(\boldsymbol{0},0)\}} \frac{\int_0^\ell J_2(w_1'')^2 + J_1(w_2'')^2 + A(w_3')^2 + \frac{J_t}{2}(\theta')^2\,\mathrm{d}x_3}{2A\int_0^\ell (w_1')^2 + (w_2')^2 + \theta^2\,\mathrm{d}x_3}, \tag{7}$$

where $A_\sharp = H^2_\sharp(0,\ell;\mathbb{R}^2) \times H^1_\sharp(0,\ell) \times H^1_\sharp(0,\ell)$ with

$$H^2_{dn}(0,\ell) := \big\{f \in H^2(0,\ell)\colon\ f(0) := 0,\ f'(0) = 0\big\}, \qquad H^2_{dd}(0,\ell) := \big\{f \in H^2_{dn}(0,\ell)\colon\ f(\ell) = 0,\ f'(\ell) = 0\big\}.$$

From (7), by means of standard Poincare's inequalities, we arrive at (3). The statements contained in (4) are a direct consequence of the assumption $\sup_\varepsilon \|\boldsymbol{E}^\varepsilon(\boldsymbol{v}^\varepsilon)\|_{L^2} < +\infty$. The characterization of $E_{\alpha 3}$, proved under the assumption that $\omega$ is simply connected, follows from a compatibility equation between infinitesimal strain and infinitesimal rotation.

The proof of the strong convergence statement (5) is quite delicate and it is achieved in several steps. First the function $\boldsymbol{v}^\varepsilon$ is extended, by using a method of [17], to the infinite cylinder $\omega \times (-\infty,+\infty)$ in such a way that $\|\boldsymbol{E}^\varepsilon(\boldsymbol{v}^\varepsilon)\|_{L^2(\omega\times(-\infty,+\infty))} \leqslant C\|\boldsymbol{E}^\varepsilon(\boldsymbol{v}^\varepsilon)\|_{L^2(\Omega)}$. Then, by mollifying $\varepsilon\nabla^\varepsilon\boldsymbol{v}^\varepsilon$ with respect to $x_3$ and by integrating over $\omega$, a function $\boldsymbol{H}^\varepsilon = \boldsymbol{H}^\varepsilon(x_3)$ is defined. An argument based on the invariance of Korn's constant under homothetic scaling (see [10,9]) yields a bound on the oscillation of $\varepsilon\nabla^\varepsilon\boldsymbol{v}^\varepsilon$ which, after appropriate estimates, leads to $\|(\boldsymbol{H}^\varepsilon)'\|_{L^2(0,\ell)} \leqslant C\|\boldsymbol{E}^\varepsilon(\boldsymbol{v}^\varepsilon)\|_{L^2(\omega\times(-\infty,+\infty))}$ and $\|\boldsymbol{H}^\varepsilon - \varepsilon\nabla^\varepsilon\boldsymbol{v}^\varepsilon\|^2_{L^2} \leqslant \varepsilon C\|\boldsymbol{E}^\varepsilon(\boldsymbol{v}^\varepsilon)\|^2_{L^2} \to 0$. From these estimates we deduce that, up to a subsequence, $\boldsymbol{H}^\varepsilon \overset{H^1}{\rightharpoonup} \boldsymbol{W}$ and that $\boldsymbol{W}$ is also the strong $L^2$-limit of $\varepsilon\nabla^\varepsilon\boldsymbol{v}^\varepsilon$.

The detailed proofs of the results presented in this Note will be given in a forthcoming paper [20]. The arguments presented can also be used to prove similar results for thin-walled beams [7,8], and for plates [3,19,21–23].


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