C. R. Acad. Sci. Paris, Ser. I 350 (2012) 745–748

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Lie Algebras/Mathematical Physics

The explicit equivalence between the standard and the logarithmic star product for Lie algebras, II

Une équivalence explicite entre les produit-étoilés standard et logarithmique pour une algèbre de Lie, II

Carlo A. Rossi

MPIM Bonn, Vivatsgasse 7, 53111 Bonn, Germany

ARTICLE INFO

Article history:
Received 5 June 2012
Accepted after revision 6 August 2012
Available online 25 September 2012

Presented by Michèle Vergne

ABSTRACT

We give a detailed proof of Rossi (2012) [10, Theorem 3.3] and comment on its nature and its relationship with the Grothendieck–Teichmüller group.

RÉSUMÉ

On donne la démonstration détaillée de Rossi (2012) [10, Théorème 3.3] ; après, on commente la nature de ce résultat and sa relation avec le groupe de Grothendieck–Teichmüller.

1. The explicit proof of [10, Theorem 3.3] and some comments thereabout

We now proceed with the proof of [10, Theorem 3.3]; let us warn the reader that we rely on [10, Digressions A, B], to which we refer frequently.

The identity on the left-hand side of [10, (2)] has been proved in [8, Section 8.3] by means of the compatibility between cup products; a different proof has been presented in [3, Section 3.2]. We will adopt the strategy proposed in [3, Section 3.2], to which we refer for more details, to prove the identity on the right-hand side.

Let us momentarily re-introduce the formal parameter $\hbar$, and consider the corresponding $\hbar$-formal Poisson bivector $\hbar\pi$.

To $\mathfrak{g}$, we may attach two natural quadratic algebras, A and $B = \wedge(\mathfrak{g}^*)$: observe that, in the present framework, A is concentrated in degree 0, while B is non-negatively graded. It is well known that A and B are Koszul algebras and are Koszul dual to each other.

We consider then the (graded) algebras $A_\hbar$, $B_\hbar$ over the ring $\mathbb{K}[[\hbar]]$. With the formal Poisson structure $\hbar\pi$, we associate the product $\star_{\log,\hbar}$ *via* the formality quasi-isomorphism $\mathcal{U}^{\log}$.

On the other hand, we may consider the $\hbar$-formal Fourier dual quadratic vector field $\hbar\widehat{\pi} = \hbar f_{ij}^k \theta_i \theta_j \partial_{\theta_k}$, borrowing previous notation for the graded basis $\{\theta_i\}$ of B and f_{ij}^k being the structure constants of $\mathfrak{g}$, on $B_\hbar$ ($\hbar\widehat{\pi}$ is the $\hbar$-shifted Chevalley–Eilenberg differential $d_\hbar$ on $B_\hbar$). Thus, the triple $(B_\hbar, d_\hbar, \wedge)$ is a dg algebra over $\mathbb{K}[[\hbar]]$: the graded formality quasi-isomorphism $\mathcal{V}$ in [5, Appendix A] admits a logarithmic version $\mathcal{V}^{\log}$ simply by replacing everywhere ω by $\omega_{\log}$, and the MC element $\mathcal{V}^{\log}(\hbar\widehat{\pi})$ endows $B_\hbar$ with the A_∞-structure over $\mathbb{K}[[\hbar]]$ given by $\wedge + \mathcal{V}^{\log}(\hbar\widehat{\pi})$. Degree arguments and the fact that the logarithmic integral weight $\varpi_\Gamma^{\log}$ associated to the graph Γ depicted in Fig. 1(i), is trivial yield that the only

E-mail address: crossi@mpim-bonn.mpg.de.

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.09.003

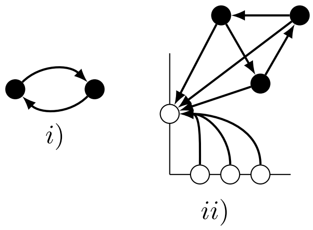

Fig. 1. (i) The unique admissible graph of type $(2,0)$ and two directed edges; (ii) an admissible graph Γ of type $(3,1,3)$ contributing to $\mathrm{L}_A^{1,\log}$.

Fig. 1. (i) Le seul graphe admissible de type $(2,0)$ avec deux arêts; (ii) un graph admissible Γ de type $(3,1,3)$ contribuant à $\mathrm{L}_A^{1,\log}$.

non-trivial Taylor components of the aforementioned A_∞-structure are $d_\hbar$ and $\wedge$, thus deformation quantization produces out of the graded commutative algebra $(B,\wedge)$ the $\hbar$-shifted Chevalley–Eilenberg complex $(B_\hbar, d_\hbar, \wedge)$. The computation of the said weight $\varpi_\Gamma^{\log}$ has been performed in [2, Lemma 6.8]: it follows from [10, Theorem 3.2, Digression A].

We regard A and B as unital A_∞-algebras: in [4, Section 6.2] a non-trivial A_∞-A-B-bimodule structure on $K=\mathbb{K}$ has been explicitly constructed, which restricts to the standard augmentation A- and B-module structure and such that $\mathrm{L}_A : A \to \underline{\mathrm{End}}_{B^+}(K)$ is an A_∞-quasi-isomorphism. Observe that we may also consider A_∞-algebras and A_∞-bimodules over the ring $K[[\hbar]]$, and all previous definitions and constructions apply to this setting as well. In particular, we may regard $(A_\hbar, \star_{\log,\hbar})$ and $(B_\hbar, d_\hbar, \wedge)$ as A_∞-algebras over $\mathbb{K}[[\hbar]]$.

Lemma 1.1. *There exists an $\hbar$-formal flat deformation $K_\hbar = \mathbb{K}[[\hbar]]$ of the A_∞-A-B-bimodule K as an A_∞-$(A_\hbar, \star_{\log,\hbar})$-$(B_\hbar, d_\hbar, \wedge)$-bimodule.*

Sketch of proof of Lemma 1.1. Let us consider the first quadrant $Q^{+,+}$ in $\mathbb{C}$, with which we associate the configuration space $C_{2,0,0}^+$ of two distinct points in $Q^{+,+}$ *modulo* rescalings. We define on $C_{2,0,0}^+$ the closed, complex-valued 1-form

$$\omega_{\log}^{+,-}(z_1,z_2) = \frac{1}{2\pi i} d\log\left(\frac{z_1-z_2}{\bar z_1 - z_2}\frac{\bar z_1 + z_2}{z_1+z_2}\right) - \frac{1}{\pi i} d\log\left(\left|\frac{\bar z_1+z_2}{z_1+z_2}\right|\right). \tag{1}$$

We denote by $\bar C_{2,0,0}^+$ the compactified version *à la* Fulton–MacPherson of $C_{2,0,0}^+$, see [6, Section 3.1] for a complete description thereof and of its boundary stratification. The 1-form (1) has the following properties:

(i) when both arguments of $\omega_{\log}^{+,-}$ approach $\mathbb{R}^+$, resp. $i\mathbb{R}^+$, $\omega_{\log}^{+,-} = \omega_{\log}^-$, resp. $\omega_{\log}^{+,-} = \omega_{\log}^+$, where $\omega_{\log}^+ = \omega_{\log}$ and $\omega_{\log}^- = \sigma^*(\omega_{\log})$, σ being the involution of $C_{2,0}^+$ given by $(z_1,z_2)\mapsto(z_2,z_1)$;

(ii) $\omega_{\log}^{+,-}$ vanishes, when its initial, resp. final point, approaches $i\mathbb{R}^+$, resp. $\mathbb{R}^+$, or the origin;

(iii) $\omega_{\log}^{+,-}$ has a simple pole of order 1 along the boundary stratum $S^1\times C_{1,0,0}^+$ of $\bar C_{2,0,0}^+$ corresponding to the collapse of its two arguments to a single point in $Q^{+,+}$, and the S^1-piece of the corresponding regularization (see [2, Section 2.1.1] or [10, Digression A]) equals the normalized volume form of S^1. There is also a $C_{1,0,0}^+$-piece in the regular part, whose presence justifies the fact that the logarithmic counterpart of the formality result for two branes requires admissible graphs with short loops, see also later on.

Properties (i)–(iii) can be checked by direct computations using local coordinates as in the proof of [4, Lemma 5.4], with due modifications because of the pole.

The explicit formulæ for the A_∞-bimodule structure $d_{K_\hbar}^{k,l}$ on $K_\hbar$ can be obtained from the ones for the corresponding A_∞-structure on $K_\hbar$ constructed in [4, Section 7] by replacing $\omega^{+,-}$ by its logarithmic counterpart (1): also for later computations, we frequently and implicitly refer to [4, Section 7].

The convergence of the logarithmic integral weights appearing in the A_∞-bimodule structure, as well as the A_∞-property itself, will be shown in a forthcoming paper in their full generality: still, we refer to the arguments in [10, Digression A] for a sketch of the proof of the convergence of the logarithmic integral weights (see also [2, Proposition 4.2]), while the L_∞-property follows by means of [10, Theorem 3.2, Digression A] (see [1, Theorem 1.8]), with some due modifications which arise from the regular parts of the 4-colored logarithmic propagators involved.

Observe that, in view of the properties of (1), if we set $\hbar=0$, we recover the A_∞-A-B-bimodule structure on K from [3]. $\square$

In particular, [3, Proposition 2.4, Lemma 2.5 and Theorem 2.7] are valid in their full generality in the logarithmic framework as well: thus, there is an algebra isomorphism

$$\mathrm{L}_{A_\hbar}^{1,\log} : (A_\hbar, \star_{\log,\hbar}) \to \mathrm{T}(\mathfrak{g})/\big(\mathrm{T}(\mathfrak{g})\otimes\langle x_i\otimes x_j - x_j\otimes x_i - \hbar[x_i,x_j]\colon i,i=1,\ldots,d\rangle\otimes\mathrm{T}(\mathfrak{g})\big)[[\hbar]] = \big(\mathrm{U}_\hbar(\mathfrak{g}),\cdot\big), \tag{2}$$

where $\mathrm{U}_\hbar(\mathfrak{g})$ is the UEA of the $\hbar$-shifted Lie algebra $\mathfrak{g}_\hbar = \mathfrak{g}$, with Lie bracket $\hbar[\bullet,\bullet]$.

The algebra isomorphism (2) is a particular case of the logarithmic version of Shoikhet's conjecture [11] about deformation quantization with generators and relations.

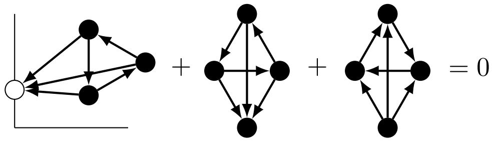

Fig. 2. Graphical representation of the relation between weights of wheel-like graphs.
Fig. 2. Représentation graphique de la relation entre les poids des graphes de type roue.

By degree reasons we may set $\hbar = 1$ in (2), hence $\mathrm{L}_A^{1,\log}$ yields an algebra isomorphism from $(A, \star_{\log})$ to $(\mathrm{U}(\mathfrak{g}), \cdot)$.

Lemma 1.2. *The algebra isomorphism $\mathrm{L}_A^{1,\log}$ equals $\mathcal{I}_{\log}$.*

Sketch of the proof of Lemma 1.2. The quasi-isomorphism $\mathrm{L}_A^{1,\log}$ is explicitly given by the formula

$$\mathrm{L}_A^{1,\log}(a_1)^m(1, b_1, \ldots, b_m) = d_K^{1,m}(a_1, 1, b_1, \ldots, b_m),$$

where one must think of $d_K^{1,m}$ as of $d_{K_\hbar}^{1,m}$ for $\hbar = 1$, and a_1 in A, and b_i in B, $i = 1, \ldots, m$.

Using the graphical definition of the deformed A_∞-A-B-bimodule structure specified by the Taylor components $d_K^{k,l}$ in [4, Sections 6.2, 7.1 and 7.2], we consider an admissible graph Γ of type $(n, 1, m)$ with n vertices of the first type (*i.e.* in $Q^{+,+}$), 1 vertex of the second type on $i\mathbb{R}^+$ and m ordered vertices of the second type on $\mathbb{R}$; Γ may have short loops at vertices of the first type, no edge may depart from the vertex on $i\mathbb{R}^+$ and no edge may arrive at a vertex on $\mathbb{R}$. From any vertex of the first type depart exactly two directed edges, whence $|E(\Gamma)| = 2n + m$.

The multidifferential operator associated to an admissible graph Γ of type $(n, 1, m)$ as before is analogous to the one appearing in the construction of [10, (1)]: b_i in B, $i = 1, \ldots, m$, is regarded as a polyderivation with constant coefficients on A, while a short loop corresponds to the divergence operator on $T_{\mathrm{poly}}(X)$ w.r.t. the constant volume form on X.

The corresponding integral weight $\varpi_\Gamma^{\log,+,-}$ is obtained by associating with an edge between two distinct vertices, resp. a short loop at a vertex of the first type, the closed 1-form (1) on $C_{2,0,0}^+$, resp. the exact 1-form $\mathrm{d}\arg(\bullet)/(4\pi)$ on $C_{1,0,0}^+$: then one integrates the corresponding closed form $\omega_\Gamma^{\log,+,-}$ of degree $2n + m$ over $C_{n,1,m}^+$. By means of the results sketched in [10, Digression A], $\omega_\Gamma^{\log,+,-}$ extends to a complex-valued, real analytic closed form of top degree on $\bar{C}_{n,1,m}^+$, whence $\varpi_\Gamma^{\log,+,-}$ converges.

The very same arguments of [3, Section 3.2] imply that a general admissible graph Γ of type $(n, 1, m)$ is the disjoint union of wheel-like graphs with spokes pointing towards the unique vertex on $i\mathbb{R}^+$ and a graph with single edges starting from ordered vertices on $\mathbb{R}$ and hitting the unique vertex on $i\mathbb{R}^+$, as in Fig. 1(ii).

The behavior of (1) along the boundary strata of $\bar{C}_{2,0,0}^+$ implies that the multidifferential operators associated with admissible graphs with no vertices of the first type contribute to the symmetrization isomorphism PBW from A to $\mathrm{U}(\mathfrak{g})$, see also [3, Section 4.2].

On the other hand, the wheel-like graphs sum up to yield exactly the invertible differential operator of infinite order and with constant coefficients specified by $j_\Gamma(\bullet)$ in $\hat{S}(\mathfrak{g}^*)$, once we have computed the logarithmic integral weights of the wheel-like graphs. For this, we use the strategy adopted in [12, Appendix B], where we replace $\omega^{+,-}$ by its logarithmic counterpart $\omega_{\log}^{+,-}$.

More precisely, the discussion in the first part of [10, Digression A] implies that [10, Theorem 3.2, Digression A], applies to differential forms associated with a wheel-like graphs with $n+1$ vertices as before. The boundary conditions for $\omega_{\log}^{+,-}$ and the regularization morphism imply the graphical relation among logarithmic integral weights depicted in Fig. 2 for $n = 3$. Observe that we have adopted the lazy convention for the signs: however, signs behave exactly as in [12, Appendix B], because the regularization morphism does not alter signs.

The second and third integral weights have to be understood w.r.t. the logarithmic propagator $\omega_{\log}$ and on configuration spaces $\bar{C}_{n+1,0}^+$ of distinct points in $\mathbb{H}^+$.

The integral weight of a wheel-like graph as in the third term has been computed explicitly in [9, Appendix A]; it remains to prove that the integral weight of a wheel-like graph as in the second term vanishes. This is, in turn, a consequence of a more general Vanishing Lemma for logarithmic integral weights.

Lemma 1.3. *Let Γ be an admissible graph of type (n, m) with $|E(\Gamma)| = 2n + m - 2$, $n \geqslant 2$, admitting a vertex of the first type which is the starting point of no edge.*

Then, the corresponding logarithmic integral weight $\varpi_\Gamma^{\log}$ vanishes.

Proof. We may safely assume, because of the standard dimensional argument, that the vertex v of the first type from which no edge departs is the endpoint of $l \geqslant 2$ edges.

Observe first that $\omega_{\log}$ depends holomorphically on its second argument. Further, since $n \geqslant 2$, we may *e.g.* fix to i the coordinate corresponding to some vertex $v_1 \neq v$ of the first type; let us denote by z_v the coordinate corresponding to the vertex v.

Then, the differential form $\omega_{\Gamma}^{\log}$ vanishes, because it depends only holomorphically on z_v and there is no non-trivial form of top degree on $\bar{C}_{n,m}^{+}$ which is holomorphic in one complex coordinate. □

Lemma 1.3 obviously applies to a wheel-like graph as in the second term of Fig. 2, and, because of the previous computations, yields the claim of Lemma 1.2. □

Lemma 1.2 is proved, and this, in turn, yields the desired claim about the explicit expression for the algebra isomorphism relating the logarithmic star product $\star_{\log}$ on A and the associative product on $\mathrm{U}(\mathfrak{g})$.

1.1. Some final comments

As remarked before the proof of Lemma 1.3, the computation of the weights corresponding to the contributions coming from wheel-like graphs is found in [9, Appendix A], whence we deduce both expressions [10, (3)] and [10, (4)] in the form of exponentials of convergent power series, whose coefficients depend on $\zeta(\bullet)$. Furthermore, it is well known that c_1 is a derivation of both $\star$ and $\star_{\log}$ on A. Therefore, by suitably changing the first coefficient, the invertible differential operators in [10, (3)] and [10, (4)] are obtained (up to the coefficient of c_1, which may be chosen freely) from the functions

$$\sqrt{j(z)} = \sqrt{\frac{1-e^{-z}}{z}} = \exp\left(-\frac{1}{4}z + \sum_{n\geqslant 1}\frac{\zeta(2n)}{(2n)(2\pi i)^n}z^{2n}\right) = \exp\left(-\frac{1}{4}z + \sum_{n\geqslant 1}\frac{B_{2n}}{(4n)(2n)!}z^{2n}\right),$$

$$\frac{1}{\Gamma(1+\frac{z}{2\pi i})} = \exp\left(\gamma z + \sum_{n\geqslant 1}\frac{\zeta(n)}{n(2\pi i)^n}z^n\right) = \sqrt{j(z)}\exp\left(\sum_{n\geqslant 1}\frac{\zeta(2n+1)}{(2n+1)(2\pi i)^{2n+1}}z^{2n+1}\right),$$

where γ denotes the Euler–Mascheroni constant, and B_n, $n \geqslant 2$, denotes the n-th Bernoulli number. Observe that the constant γ appears mainly for aesthetical reasons.

Therefore, a bit improperly, we may use the reciprocal Γ-function with shifted argument to construct the isomorphism $\mathcal{I}_{\log}$: in fact, up to the term of first order (whose coefficient may be chosen freely because c_1 is a derivation for both products $\star$ and $\star_{\log}$), the exponential of the power series in [10, (4)] coincides with the function first considered in [7, Section 4.6] in a discussion about incarnations of the Grothendieck–Teichmüller (GRT for short) group. The very same expression has been re-discovered in [9, Section 4.9] in the framework of exotic L_∞-automorphisms of $T_{\mathrm{poly}}(X)$ and their connection with the GRT group.

As an immediate corollary of [10, Theorem 3.3], for a Lie algebra $\mathfrak{g}$ as in Section 1, the star products in [10, (1)] on A are equivalent w.r.t. the invertible differential operator with constant coefficients and of infinite order associated with the element of $\widehat{\mathrm{S}}(\mathfrak{g})$ given by

$$\exp\left(\sum_{n\geqslant 1}\frac{\zeta(2n+1)}{(2n+1)(2\pi i)^{2n+1}}c_{2n+1}(x)\right), \quad x \in \mathfrak{g}.$$

Let us remark that a more conceptual approach to the Lie algebra $\mathfrak{grt}$ of the GRT group in the framework of deformation quantization can be found in the fundamental paper [13], in particular in [13, Sections 7.4, 7.5], where modifications of the Duflo element *via* elements of $\mathfrak{grt}$ have been discussed in details.

References

[1] Anton Alekseev, Johannes Löffler, Carlo A. Rossi, Charles Torossian, Stokes' Theorem in presence of poles and logarithmic singularities, 2012, in preparation.
[2] Anton Alekseev, Johannes Löffler, Carlo A. Rossi, Charles Torossian, The logarithmic formality quasi-isomorphism, 2012, in preparation.
[3] Damien Calaque, Giovanni Felder, Deformation quantization with generators and relations, J. Algebra 337 (2011) 1–12, http://dx.doi.org/10.1016/j.jalgebra.2011.03.037.
[4] Damien Calaque, Giovanni Felder, Andrea Ferrario, Carlo A. Rossi, Bimodules and branes in deformation quantization, Compos. Math. 147 (1) (2011) 105–160, http://dx.doi.org/10.1112/S0010437X10004847.
[5] Alberto S. Cattaneo, Giovanni Felder, Relative formality theorem and quantisation of coisotropic submanifolds, Adv. Math. 208 (2) (2007) 521–548, http://dx.doi.org/10.1016/j.aim.2006.03.010.
[6] Alberto S. Cattaneo, Carlo A. Rossi, Charles Torossian, Biquantization of symmetric pairs and the quantum shift, arXiv:1105.5973, 2011.
[7] Maxim Kontsevich, Operads and motives in deformation quantization, Moshé Flato (1937–1998), Lett. Math. Phys. 48 (1) (1999) 35–72, http://dx.doi.org/10.1023/A:1007555725247.
[8] Maxim Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (3) (2003) 157–216.
[9] Sergei Merkulov, Exotic automorphisms of the Schouten algebra of polyvector fields, arXiv:0809.2385, 2008.
[10] Carlo A. Rossi, The explicit equivalence between the standard and the logarithmic star product for Lie algebras, I, C. R. Acad. Sci. Paris, Ser. I 350 (13–14) (2012) 661–664, http://dx.doi.org/10.1016/j.crma.2012.08.001.
[11] Boris Shoikhet, Kontsevich formality and PBW algebras, arXiv:0708.1634, 2007.
[12] Thomas Willwacher, A counterexample to the quantizability of modules, Lett. Math. Phys. 81 (3) (2007) 265–280.
[13] Thomas Willwacher, M. Kontsevich's graph complex and the Grothendieck–Teichmüller Lie algebra, arXiv:1009.1654, 2010.