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On the compatibility between cup products, the Alekseev–Torossian connection and the Kashiwara–Vergne conjecture, I

Compatibilité entre cup-produits, connexion d'Alekseev–Torossian et conjecture de Kashiwara–Vergne, I

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ABSTRACT

For a finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}\supset\mathbb{C}$, we deduce from the compatibility between cup products (Kontsevich, 2003, [6, Section 8]), on the one hand, and from the Kashiwara–Vergne conjecture (Kashiwara and Vergne, 1978, [4]), on the other hand, alternative ways of re-writing Kontsevich product $\star$ on $\mathrm{S}(\mathfrak{g})$.
In this first part, we fix notation and conventions, revise the main features of Kontsevich's star product and examine the Kashiwara–Vergne conjecture and its relationship with Kontsevich's star product.

RÉSUMÉ

Soit $\mathfrak{g}$ une algèbre de Lie de dimension finie sur un champ $\mathbb{K}\supset\mathbb{C}$. Du résultat (Kontsevich, 2003, [6, Section 8]) sur la compatibilité entre cup produits et de la conjecture de Kashiwara–Vergne (Kashiwara et Vergne, 1978, [4]), on déduit deux écritures alternatives du produit-étoilé de Kontsevich sur $\mathrm{S}(\mathfrak{g})$.
Dans cette première partie, on fixe la notation et les conventions; on rappelle le produit-étoilé de Kontsevich, on discute la conjecture de Kashiwara–Vergne et la relation entre elle et le produit-étoilé de Kontsevich.

1. Introduction

For a general finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}\supset\mathbb{C}$, we consider the symmetric algebra $A=\mathrm{S}(\mathfrak{g})$. Deformation quantization *à la* Kontsevich [6] endows A with an associative, non-commutative product $\star$.

In this short note, we deduce a way of re-writing the product $\star$ on A in terms of the Lie series F, G appearing in the combinatorial Kashiwara–Vergne (shortly, from now on, KV) conjecture [4]. In fact, we prove a similar claim by deducing it from the compatibility between cup products for Kontsevich's formality quasi-isomorphism [6, Section 8]. Both claims are proved in a constructive way (see later on Identity (8) and [8, Identity (7)]).

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Remark 1.1. The results of [8, Sections 1, 2] were already somehow present in the seminal work [9] of C. Torossian: we present them here in a different fashion, and the compatibility between cup products in 0-th cohomology is proved in a different way than in [6, Subsection 8.2].

Remark 1.2. The conjecture of Kashiwara–Raïs–Vergne in the framework of point-supported distributions on Lie groups and Lie algebras has been proved in [7, Section 4] as a consequence of a result extending the compatibility between cup products from [6, Subsection 8.2] to an A_∞-algebra quasi-isomorphism between poly-vector fields and poly-differential operators over $X=\mathbb{K}^d$.

However, the main result [7, Identity (55)] does not hold true, because [6, Kontsevich's Vanishing Lemma 6.6] does not hold true in general for the compactified configuration spaces considered in [7, Subsection 2.3]. This has been proved by direct computations either in [1, Example 2] or in [2, Subsection 3.4].

Still, the proof of the KRV conjecture in [7, Section 4] remains true, as it makes use only of compatibility between cup products. The techniques presented here for the proof of compatibility between cup products are more general than the ones adopted in [7] and imply the results of [2] as well, upon the choice of certain homology chains in the relevant (compactified) configuration spaces.

2. Notation and conventions

We consider a field $\mathbb{K}\supset\mathbb{C}$.

We denote by $\mathfrak{g}$ a finite-dimensional Lie algebra over $\mathbb{K}$ of dimension d; by $\{x_i\}$ we denote a $\mathbb{K}$-basis of $\mathfrak{g}$. With $\mathfrak{g}$ we associate the linear Poisson variety $X=\mathfrak{g}^*$ over $\mathbb{K}$ endowed with the Kirillov–Kostant Poisson bivector field.

3. Kontsevich's star product

Let $X=\mathbb{K}^d$ and $\{x_i\}$ a system of global coordinates on X, for $\mathbb{K}$ as above.

For a pair (n,m) of non-negative integers, by $\mathcal{G}_{n,m}$ we denote the set of admissible graphs of type (n,m): an element Γ of $\mathcal{G}_{n,m}$ is a directed graph with n, resp. m, vertices of the first, resp. second type, such that i) there is no directed edge departing from any vertex of the second type and ii) Γ admits whether multiple edges nor short loops (*i.e.* given two distinct vertices v_i, $i=1,2$, of Γ there is at most one directed edge from v_1 to v_2 and there is no directed edge, whose endpoint coincides with the initial point). By $E(\Gamma)$ we denote the set of edges of Γ in $\mathcal{G}_{n,m}$.

We denote by $C^+_{n,m}$ the configuration space of n points in the complex upper half-plane $\mathbb{H}^+$ and m ordered points on the real axis $\mathbb{R}$ *modulo* the componentwise action of rescalings and real translations: provided $2n+m-2\geqslant 0$, $C^+_{n,m}$ is a smooth, oriented manifold of dimension $2n+m-2$. We denote by $\bar{C}^+_{n,m}$ its compactification *à la* Fulton–MacPherson introduced in [6, Section 5]: $\bar{C}^+_{n,m}$ is a compact, oriented, smooth manifold with corners of dimension $2n+m-2$.

We denote by ω the closed, real-valued 1-form

$$\omega(z_1,z_2)=\frac{1}{2\pi}\,\mathrm{d}\arg\left(\frac{z_1-z_2}{\bar{z}_1-z_2}\right),\quad (z_1,z_2)\in\left(\mathbb{H}^+\sqcup\mathbb{R}\right)^2,\ z_1\neq z_2,$$

where $\arg(\bullet)$ denotes the $[0,2\pi)$-valued argument function on $\mathbb{C}\smallsetminus\{0\}$ such that $\arg(i)=\pi/2$; ω extends to a smooth, closed 1-form on $\bar{C}^+_{2,0}$, such that i) when the two arguments approach to each other in $\mathbb{H}^+$, ω equals the normalized volume form vol_{S^1} on S^1 and ii) when the first argument approaches $\mathbb{R}$, ω vanishes.

We introduce $T_{\mathrm{poly}}(X)=A[\theta_1,\dots,\theta_d]$, $A=C^\infty(X)$, where $\{\theta_i\}$ denotes a set of graded variables of degree 1 dual to $\{x_i\}$, which commute with A and anticommute with each other (one may think of θ_i as ∂_i with a shifted degree). We further consider the well-defined linear endomorphism τ of $T_{\mathrm{poly}}(X)^{\otimes 2}$ of degree -1 defined *via*

$$\tau=\partial_{\theta_i}\otimes\partial_{x_i}.$$

With Γ in $\mathcal{G}_{n,m}$ such that $|E(\Gamma)|=2n+m-2$, γ_i, $i=1,\dots,n$, elements of $T_{\mathrm{poly}}(X)$ and a_j, $j=1,\dots,m$, elements of A, we associate a map *via*

$$\big(\mathcal{U}_\Gamma(\gamma_1,\dots,\gamma_n)\big)(a_1\otimes\cdots\otimes a_m)=\mu_{m+n}\Bigg(\int\limits_{\bar{C}^+_{n,m}}\omega_{\tau,\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_n\otimes a_1\otimes\cdots\otimes a_m)\Bigg),$$

$$\omega_{\tau,\Gamma}=\prod_{e\in E(\Gamma)}\omega_{\tau,e},\quad \omega_{\tau,e}=\pi_e^*(\omega)\otimes\tau_e,\tag{1}$$

τ_e being the graded endomorphism of $T_{\mathrm{poly}}(X)^{\otimes(m+n)}$ which acts as τ on the two factors of $T_{\mathrm{poly}}(X)$ corresponding to the initial and final point of the edge e, and μ_{m+n} denotes the multiplication map from $T_{\mathrm{poly}}(X)^{m+n}$ to $T_{\mathrm{poly}}(X)$, followed by the natural projection from $T_{\mathrm{poly}}(X)$ onto A by setting $\theta_i=0$, $i=1,\dots,d$. By π_e we denote the natural projection from

$C_{n,m}^+$ onto $C_{2,0}^+$ or $C_{1,1}^+$ associated with the edge e. We may re-write (1) by splitting the form-part and the poly-differential operator part as

$$\big(\mathcal{U}_\Gamma(\gamma_1,\dots,\gamma_n)\big)(a_1\otimes\cdots\otimes a_m)=\varpi_\Gamma\big(\mathcal{B}_\Gamma(\gamma_1,\dots,\gamma_n)\big)(a_1,\dots,a_m),\quad \varpi_\Gamma=\int\limits_{\overline{C}_{n,m}^+}\omega_\Gamma.$$

In [6, Theorem 6.4], the following theorem has been proved

Theorem 3.1. *For a Poisson bivector field π on X, and a formal parameter $\hbar$, the formula*

$$f_1\star_\hbar f_2=\sum_{n\geqslant 0}\frac{\hbar^n}{n!}\sum_{\Gamma\in\mathcal{G}_{n,2}}\big(\mathcal{U}_\Gamma(\underbrace{\pi,\dots,\pi}_{n})\big)(f_1,f_2),\quad f_i\in A,\ i=1,2, \tag{2}$$

defines a $\mathbb{K}_\hbar=\mathbb{K}[\![\hbar]\!]$-linear, associative product on $A_\hbar=A[\![\hbar]\!]$.

We now state the main result of this note, whose proof will be given in full detail in [8, Section 1].

Let us consider a piecewise differentiable curve γ on $\overline{C}_{2,0}^+$ connecting a given point in $\overline{C}_2=S^1$ with $\overline{C}_{0,2}^+=\{0,1\}$ and whose interior is in $C_{2,0}^+$.

Theorem 3.2. *For a Lie algebra $\mathfrak{g}$ as in Section 2, there exist a smooth 0-form $\mathcal{T}^\pi$ with values in the bidifferential operators on $A=\mathrm{S}(\mathfrak{g})$ and smooth 1-forms Ω_i^π on $C_{2,0}^+$, $i=1,2$, with values in $\mathfrak{g}\otimes\widehat{\mathrm{S}}(\mathfrak{g}^*)^{\otimes 2}$, such that the following identity holds true*:

$$f_1\star f_2-f_1f_2=\int\limits_\gamma\big(\mathcal{T}^\pi\big(\Omega_1^\pi\big([\pi,f_1],f_2\big)\big)+\mathcal{T}^\pi\big(\Omega_2^\pi\big(f_1,[\pi,f_2]\big)\big)\big),\quad f_i\in A,\ i=1,2. \tag{3}$$

The explicit construction of $\mathcal{T}^\pi$ and Ω_i^π, $i=1,2$, will be given in full detail in [8, Section 1].

4. The KV conjecture and Kontsevich's star product

Given $\mathfrak{g}$ as in Section 2, the KV conjecture states the existence of two Lie series F, G, which are convergent in a neighborhood U of $(0,0)$ in $\mathfrak{g}\times\mathfrak{g}$, which satisfy the two identities

$$y_1+y_2-\log\big(e^{y_2}e^{y_1}\big)=\big(1-e^{-\mathrm{ad}(y_1)}\big)F(y_1,y_2)+\big(e^{\mathrm{ad}(y_2)}-1\big)G(y_1,y_2), \tag{4}$$

$$\begin{aligned}&\mathrm{tr}_\mathfrak{g}\big(\mathrm{ad}(y_1)\partial_{y_1}F(y_1,y_2)\big)+\mathrm{tr}_\mathfrak{g}\big(\mathrm{ad}(y_2)\partial_{y_2}G(y_1,y_2)\big)\\&\quad=\frac{1}{2}\mathrm{tr}_\mathfrak{g}\left(\frac{\mathrm{ad}(y_1)}{e^{\mathrm{ad}(y_1)}-1}+\frac{\mathrm{ad}(y_2)}{e^{\mathrm{ad}(y_2)}-1}-\frac{\mathrm{ad}(Z(y_1,y_2))}{e^{\mathrm{ad}(Z(y_1,y_2))}-1}-1\right),\end{aligned} \tag{5}$$

for (y_1,y_2) in U, such that the BCH Lie series $Z(y_1,y_2)=\log(e^{y_1}e^{y_2})$ converges.

We recall from [6, Subsection 8.2] the algebra isomorphism from $(A,\star)$ and the Universal Enveloping Algebra $(\mathrm{U}(\mathfrak{g}),\cdot)$ of $\mathfrak{g}$,

$$\mathcal{I}(f_1\star f_2)=\mathcal{I}(f_1)\cdot\mathcal{I}(f_2),\quad f_i\in A,\ i=1,2, \tag{6}$$

where $\mathcal{I}$ is the composition of the PBW isomorphism from A to $\mathrm{U}(\mathfrak{g})$ with the Duflo element in $\widehat{\mathrm{S}}(\mathfrak{g}^*)$,

$$\sqrt{j(x)}=\sqrt{\det_\mathfrak{g}\left(\frac{1-e^{-\mathrm{ad}(x)}}{\mathrm{ad}(x)}\right)},\quad x\in\mathfrak{g}.$$

As a corollary of (6), we have the identity

$$e^{y_1}\star e^{y_2}=D(y_1,y_2)e^{Z(y_1,y_2)},\quad D(y_1,y_2)=\frac{\sqrt{j(y_1)}\sqrt{j(y_2)}}{\sqrt{j(Z(y_1,y_2))}},\ y_i\in\mathfrak{g},\ i=1,2. \tag{7}$$

We observe that (7) has been proved by different methods in [5] and [3, Subsubsections 3.1.2–3.1.4]. Let us replace in (7) π by $t\pi$, for t in the unit interval: we write $\star_t$ for the corresponding product, whence

$$e^{y_1}\star_t e^{y_2}=D(ty_1,ty_2)e^{Z_t(y_1,y_2)},\quad Z_t(y_1,y_2)=\frac{Z(ty_1,ty_2)}{t}.$$

It follows directly from (2) that $e^{y_1}\star_1 e^{y_2}=e^{y_1}\star e^{y_2}$ and $e^{y_1}\star_0 e^{y_2}=e^{y_1}e^{y_2}$, y_i in $\mathfrak{g}$.

Identity (4) implies that (see e.g. [4, Lemma 3.2])

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_t(y_1,y_2)=\left(\left\langle\left[y_1,F_t(y_1,y_2)\right],\partial_{y_1}\right\rangle+\left\langle\left[y_2,G_t(y_1,y_2)\right],\partial_{y_2}\right\rangle\right)Z_t(y_1,y_2),$$

where $F_t(y_1,y_2)=F(ty_1,ty_2)/t$ and similarly for $G_t(y_1,y_2)$; further, for two convergent Lie series F, G in some neighborhood of $\mathfrak{g}\times\mathfrak{g}$, we set

$$\left(\left\langle F(y_1,y_2),\partial_{y_1}\right\rangle\right)G(y_1,y_2)=\frac{\mathrm{d}}{\mathrm{d}t}G\left(y_1+tF(y_1,y_2),y_2\right)\bigg|_{t=0},$$

and similarly for $(\langle F(y_1,y_2),\partial_{y_2}\rangle)G(y_1,y_2)$.

On the other hand, combining [4, Lemma 3.2] with [4, Lemma 3.3] and (5), and observing that $\sqrt{j(\bullet)}$ is $\mathfrak{g}$-invariant, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}D(ty_1,ty_2)&=\mathrm{tr}_{\mathfrak{g}}\left(\mathrm{ad}(y_1)\partial_{y_1}F_t(y_1,y_2)\right)+\mathrm{tr}_{\mathfrak{g}}\left(\mathrm{ad}(y_2)\partial_{y_2}G_t(y_1,y_2)\right)D(ty_1,ty_2)\\&\quad+\left(\left\langle\left[y_1,F_t(y_1,y_2)\right],\partial_{y_1}\right\rangle+\left\langle\left[y_2,G_t(y_1,y_2)\right],\partial_{y_2}\right\rangle\right)D(ty_1,ty_2).\end{aligned}$$

Combining both previous results, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{y_1}\star_t e^{y_2}\right)&=\left(\left\langle\left[y_1,F_t(y_1,y_2)\right],\partial_{y_1}\right\rangle+\mathrm{tr}_{\mathfrak{g}}\left(\mathrm{ad}(y_1)\partial_{y_1}F_t(y_1,y_2)\right)\right)\left(e^{y_1}\right)\star_t e^{y_2}\\&\quad+e^{y_1}\star_t\left(\left\langle\left[y_2,G_t(y_1,y_2)\right],\partial_{y_1}\right\rangle+\mathrm{tr}_{\mathfrak{g}}\left(\mathrm{ad}(y_1)\partial_{y_1}G_t(y_1,y_2)\right)\right)\left(e^{y_2}\right).\end{aligned}$$

With the Lie series F_t, G_t, one may associate two smooth 1-forms Ω_i^{KV}, $i=1,2$, on the unit interval with values in $\mathfrak{g}\otimes\widehat{\mathrm{S}}(\mathfrak{g}^*)$ (see the computations at the beginning of [8, Section 1]), such that the following identities hold true:

$$\begin{aligned}\Omega_1^{\mathrm{KV}}\left(\left[\pi,e^{y_1}\right],e^{y_2}\right)&=\left\langle\left[y_1,F_t(y_1,y_2)\,\mathrm{d}t\right],\partial_{y_1}\right\rangle\left(e^{y_1}\right)\otimes e^{y_2}+\mathrm{tr}_{\mathfrak{g}}\left(\mathrm{ad}(y_1)\partial_{y_1}\left(F_t(y_1,y_2)\,\mathrm{d}t\right)\right)e^{y_1}\otimes e^{y_2},\\\Omega_2^{\mathrm{KV}}\left(\left[\pi,e^{y_1}\right],e^{y_2}\right)&=e^{y_1}\otimes\left\langle\left[y_2,G_t(y_1,y_2)\,\mathrm{d}t\right],\partial_{y_2}\right\rangle\left(e^{y_2}\right)+\mathrm{tr}_{\mathfrak{g}}\left(\mathrm{ad}(y_2)\partial_{y_2}\left(G_t(y_1,y_2)\,\mathrm{d}t\right)\right)e^{y_1}\otimes e^{y_2},\end{aligned}$$

whence, denoting by $\mathcal{T}_t(\bullet,\bullet)$ the t-dependent bidifferential operator of infinite order $\mathcal{T}_t(f_1,f_2)=f_1\star_t f_2$, f_i in A, we find the homotopy formula

$$f_1\star f_2-f_1f_2=\int_0^1\left(\mathcal{T}_t\left(\Omega_1^{\mathrm{KV}}\left([\pi,f_1],f_2\right)\right)+\mathcal{T}_t\left(\Omega_2^{\mathrm{KV}}\left(f_2,[\pi,f_1]\right)\right)\right),\tag{8}$$

which is similar in its structure to the homotopy formula [8, Identity (7)] obtained by deforming the product $\star$ on $C_{2,0}^+$, the forms Ω_i^{KV}, $i=1,2$, being analogous to the pull-back with respect to a chosen curve γ as in Theorem 3.2.

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Further reading

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