C. R. Acad. Sci. Paris, Ser. I 350 (2012) 519–524

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebraic Geometry

Motivic Milnor fibers of a rational function

Fibres de Milnor motiviques d'une fraction rationnelle

Michel Raibaut

Institut de mathématiques de Jussieu, 4, place Jussieu, case 247, 75252 Paris cedex 5, France

ARTICLE INFO

Article history:
Received 15 February 2012
Accepted after revision 27 April 2012
Available online 22 May 2012

Presented by Claire Voisin

ABSTRACT

Let P and Q be two complex polynomials and f be the induced rational function. In this Note we define a motivic Milnor fiber of the germ of f at an indeterminacy point x for a value a, a motivic Milnor fiber of f for a value a and finally motivic bifurcation sets.

RÉSUMÉ

Soit P et Q deux polynômes à coefficients complexes et f l'application rationnelle quotient induite. Dans cette Note nous introduisons une fibre de Milnor motivique du germe de f en un point d'indétermination x pour une valeur a, puis une fibre de Milnor motivique de f pour une valeur a et enfin des ensembles de bifurcation motiviques.

Version française abrégée

Soit d un entier positif et P et Q deux polynômes de $\mathbb{C}[x_1,\dots,x_d]$. On note $\mathcal{I}$ le lieu des zéros communs de P et Q et f l'application rationnelle P/Q bien définie de $\mathbb{A}^d_{\mathbb{C}} \setminus \mathcal{I}$ vers $\mathbb{P}^1_{\mathbb{C}}$. Gusein-Zade, Luengo et Melle Hernández étudient dans [5] le germe de f en un point d'indétermination x. Ils montrent en particulier que pour toute valeur a de $\mathbb{P}^1_{\mathbb{C}}$, il existe ϵ_0 positif, tel que pour tout ϵ positif et inférieur à ϵ_0, l'application f de $B(x,\epsilon) \setminus \mathcal{I}$ vers $\mathbb{P}^1_{\mathbb{C}}$ est une fibration topologique localement triviale au-dessus d'un disque épointé de a. La fibre de cette fibration est appelée fibre de Milnor de f au point x en la valeur a, notée $F_{x,a}$, et est munie d'une transformation de monodromie $T_{x,a}$ induite par la fibration. Une valeur a sera dite typique, et atypique dans le cas contraire, si cette fibration est triviale sur un voisinage non épointé de a. L'ensemble des valeurs atypiques est appelé ensemble de bifurcation du germe de f en x. Cet ensemble est fini. Dans [5] les auteurs donnent une formule à la A'Campo pour la fonction zêta de la monodromie $T_{x,a}$ et pour ses nombres de Lefschetz. Dans [6] ils montrent, comme dans le cas polynomial, que f est une fibration topologique localement triviale sur le complémentaire d'un sous ensemble fini de la droite projective $\mathbb{P}^1_{\mathbb{C}}$. Ils obtiennent ainsi une fibre de Milnor de f pour une valeur a, notée F_a, et munie d'une transformation de monodromie T_a, et définissent là encore un ensemble de bifurcation.

Les analogues faisceautiques de ces situations topologiques se construisent comme suit. Soit x un point d'indétermination et a une valeur de la droite projective. Soit $\mathcal{F}_a$ le complexe borné de faisceaux constructibles $\psi_{\bar{f}-a}(R\bar{i}_!\mathbb{Q}_X)$ où ψ est le foncteur cycles proches de Deligne, X est le graphe de f, $\overline{X}$ est son adhérence dans $\mathbb{C}^d \times \mathbb{P}^1_{\mathbb{C}}$, $\bar{i}$ est l'injection de X dans $\overline{X}$, $\bar{f}$ est l'extension de f à $\overline{X}$ et $\bar{f} - a$ est $1/\bar{f}$ pour a infini. On se référera au diagramme commutatif ci-dessous (Section 2). Ce complexe de faisceaux est supporté par la fibre de $\bar{f}$ en a et son germe en (x,a), noté $\mathcal{F}_{x,a}$, est le complexe

E-mail address: raibaut@math.jussieu.fr.

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.04.021

associé à la fibre de Milnor $F_{x,a}$. Celui-ci est naturellement muni d'une action de monodromie induite par la construction du faisceau des cycles proches. Comme dans le cas polynomial les groupes de cohomologies $\mathcal{H}^k(\mathcal{F}_{x,a})^{\circlearrowleft}$ sont isomorphes aux groupes de cohomologies $H_c^k(F_{x,a},\mathbb{Q})^{\circlearrowleft T_{x,a}}$ (Proposition 2.1). Cette construction existe au niveau des modules de Hodge mixtes de M. Saito [11], on obtient ainsi le complexe de modules de Hodge mixtes sur (x,a) noté $\mathcal{F}_{x,a}^{\mathrm{MHM}\circlearrowleft}$. La structure de Hodge mixte sous jacente du module de Hodge mixte $\mathcal{H}^k\mathcal{F}_{x,a}^{\mathrm{MHM}\circlearrowleft}$ est la *structure de Hodge mixte limite* de $H_c^k(F_{x,a},\mathbb{Q})^{\circlearrowleft T_{x,a}}$.

Au niveau global, on considère le complexe de faisceaux constructibles $\mathcal{G}_a$ appartenant à la catégorie dérivée $\mathcal{D}_c^b(\{a\})^{\mathrm{mon}}$ défini par $R\hat{f}_!\psi_{\hat{f}-a}(R(\hat{i}\circ i)_!\mathbb{Q}_X)$ où $(\hat{X},\hat{i},\hat{f})$ est une compactification du morphisme $(\overline{X},\overline{f})$, on se référera au diagramme commutatif ci-dessous. Ce complexe de faisceaux constructibles $\mathcal{G}_a$ ne dépend pas de la compactification $(\hat{X},\hat{i},\hat{f})$. Pour tout k, il existe un isomorphisme entre $\mathcal{H}^k\mathcal{G}_a^{\circlearrowleft}$ et $H_c^k(F_a,\mathbb{Q})^{\circlearrowleft T_a}$ (Proposition 2.2). Cette construction existe là aussi au niveau des modules de Hodge mixtes, on obtient ainsi le complexe de modules de Hodge mixtes sur $\{a\}$ noté $\mathcal{G}_a^{\mathrm{MHM}\circlearrowleft}$. La *structure de Hodge mixte limite* de $H_c^k(F_a,\mathbb{Q})^{\circlearrowleft T_a}$ est la structure sous jacente du module de Hodge mixte $\mathcal{H}^k\mathcal{G}_a^{\mathrm{MHM}\circlearrowleft}$.

Explicitons maintenant les invariants motiviques associés aux constructions précédentes. Soit $\mathcal{L}(\hat{X})$ l'espace des arcs de $\hat{X}$ et pour n et δ deux entiers strictement positifs, considérons la partie semi-algébrique $\hat{X}_n^\delta := \{\varphi\in\mathcal{L}(\hat{X}) \mid \mathrm{ord}_t\,\varphi^*(\hat{X}\setminus X)\leqslant n\delta,\ \mathrm{ord}_t[\hat{f}(\varphi(t))-\hat{f}(\varphi(0))]=n\}$. Cet espace d'arcs est muni de l'action naturelle du groupe multiplicatif $\mathbb{G}_m$ sur les arcs et du morphisme à valeurs dans $\hat{X}\times\mathbb{G}_m$ qui a un arc φ associe le couple $[\varphi(0),\mathrm{ac}(\hat{f}(\varphi)(t)-\hat{f}(\varphi)(0))]$ où $\mathrm{ac}(\hat{f}(\varphi)(t)-\hat{f}(\varphi)(0))$ désigne le premier coefficient non nul de la série $\hat{f}(\varphi)(t)-\hat{f}(\varphi)(0)$. Comme dans [9, Section 4] on considère la fonction zêta motivique globale $Z_{\hat{f},X}^\delta(T)$ de $\hat{f}$ sur l'ouvert X, égale à $\sum_{n\geqslant1}\mu(\hat{X}_n^\delta)T^n$ où μ désigne ici la mesure motivique de Kontsevich à valeurs dans un anneau de Grothendieck des variétés $\mathcal{M}_{\hat{X}\times\mathbb{G}_m}^{\mathbb{G}_m}$ (cf. [4, 2.2] et Section 3). En appliquant [4, 3.8] pour δ suffisament grand cette fonction est rationnelle et admet une limite quand T tend vers l'infini indépendante de δ. La limite $-\lim_{T\to\infty}Z_{\hat{f},X}^\delta(T)$, notée $S_{\hat{f},X}^{\mathrm{global}}$, appartient à $\mathcal{M}_{\hat{X}\times\mathbb{G}_m}^{\mathbb{G}_m}$.

La restriction de $S_{\hat{f},X}^{\mathrm{global}}$ à $\{(x,a)\}\times\mathbb{G}_m$, notée $S_{f,x,a}$, est la *fibre de Milnor motivique de f en x pour la valeur a*. L'image directe par $\hat{f}$ de $S_{\hat{f},X}^{\mathrm{global}}$ sur le produit $\mathbb{P}_{\mathbb{C}}^1\times\mathbb{G}_m$ suivi de la restriction à $\{a\}\times\mathbb{G}_m$ notée $S_{f,a}$, est appelée *fibre de Milnor motivique de f pour la valeur a*. Quelques propriétés :

- Les motifs $S_{f,x,a}$ et $S_{f,a}$ ne dépendent pas de la compactification et leur fibre en 1 se réalise sur les classes $[\mathcal{F}_{x,a}^{\mathrm{MHM}\circlearrowleft}]$ et $[\mathcal{G}_a^{\mathrm{MHM}\circlearrowleft}]$ dans leur groupe de Grothendieck respectif (Theorems 3.9 et 3.14).
- Comme dans le cas polynomial [3], la caractéristique d'Euler $Eu_c(\mu(\hat{X}_{n,x,a}^{\delta,1}))$ est le n-ième nombre de Lefschetz de la monodromie $T_{x,a}$ avec $\hat{X}_{n,x,a}^{\delta,1}$ la fibre de $\hat{X}_n^\delta$ en $(x,a,1)$ (Theorem 3.11).

Une valeur a sera alors dite *motiviquement atypique* pour f en x si $S_{f,x,a}$ est non nul, l'ensemble de ces valeurs est fini et appelé *ensemble de bifurcation motivique* de f en x. Une valeur a sera dite *motiviquement atypique* si c'est une valeur critique de f ou si $S_{f,a}$ est différent de $[f^{-1}(a)\times\{a\}\times\mathbb{G}_m]$. L'ensemble de ces valeurs est fini et appelé *ensemble de bifurcation motivique* de f (Theorems 3.6 et 3.16).

1. Introduction

Let d be a positive integer and P and Q be two polynomials in $\mathbb{C}[x_1,\dots,x_d]$. We denote by $\mathcal{I}$ the common zero set of P and Q and by f the rational function P/Q well defined on $\mathbb{A}_{\mathbb{C}}^d\setminus\mathcal{I}$ to $\mathbb{P}_{\mathbb{C}}^1$. Gusein-Zade, Luengo and Melle Hernández in [5] and [6] (in the more general meromorphic setting) studied the germ of f at an indeterminacy point x. They showed in particular that for all value a in $\mathbb{P}_{\mathbb{C}}^1$, there exists ϵ_0 positive such that for any positive ϵ smaller than ϵ_0, the function f from $B(x,\epsilon)\setminus\mathcal{I}$ to $\mathbb{P}_{\mathbb{C}}^1$ is a topological locally trivial fibration over a punctured neighborhood of a. Thus, they defined a Milnor fiber of f at x for each value a, denoted by $F_{x,a}$, endowed with a monodromy action induced by the fibration and denoted by $T_{x,a}$. They proved, as A'Campo did it in the holomorphic case, a formula for the monodromy zeta function of $T_{x,a}$ and its Lefschetz numbers, in terms of a log-resolution of the zero set of P and the zero set of Q. A value a is called typical, and atypical in the other case, if the fibration can be extended as a trivial fibration over a neighborhood of a. The set of atypical values is finite and called the bifurcation set of the germ of f at x. In [6] Gusein-Zade, Luengo and Melle Hernández studied the global situation and showed, as in the holomorphic case, that f is a topological locally trivial fibration over the complex projective line away from a finite subset. This allowed them to construct a Milnor fiber of f for the value a, denoted by F_a, which carries again a monodromy action T_a. They also defined a bifurcation set.

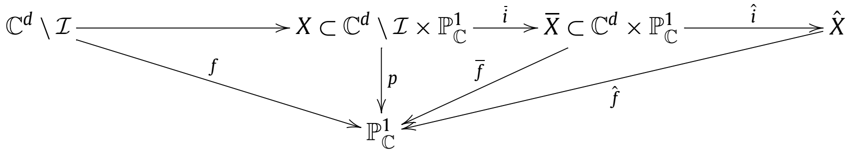
In this Note we start by giving sheaf-theoretic interpretations of these topological situations. These constructions will also be defined at the level of mixed Hodge modules of M. Saito and will give the limit mixed Hodge structure on the cohomology groups of the Milnor fibers.

In the same way as in Denef–Loeser [1] and [3] and more recently in Guibert–Loeser–Merle [4], we introduce motives associated to the above topological objects. In particular, we construct a motivic Milnor fiber of the germ of f at x for the value a and a global motivic Milnor fiber of f for the value a, taking account of singularities at infinity. As in the polynomial case [9], we obtain a motivic bifurcation set in the local and global situations. These objects contain additive and multiplicative invariants of the Milnor fibers, and as in [1] realize on the classes in appropriate Grothendieck rings of the above mixed Hodge modules. We also obtain the formula for the Lefschetz numbers of $T_{x,a}$ in [5].

This Note announces results of [10].

2. Sheaf point of view

Below, by a variety we mean a separated reduced scheme of finite type over $\mathbb{C}$. Let us consider the following commutative diagram:



where X is the graph of f in $\mathbb{C}^d \setminus \mathcal{I} \times \mathbb{P}^1_{\mathbb{C}}$, $\overline{X}$ is its Zariski closure in $\mathbb{C}^d \times \mathbb{P}^1_{\mathbb{C}}$, $\bar{i}$ is the associated closed immersion, p and $\overline{f}$ are the projections on $\mathbb{P}^1_{\mathbb{C}}$, $\hat{X}$ is a variety, $\hat{i}$ is an open and dominant immersion and $\hat{f}$ is a proper map. We will denote by i the composition of immersions $\hat{i} \circ \bar{i}$, by $\overline{F}$ the complementary of X in $\overline{X}$, and by $\hat{F}$ the complementary of X in $\hat{X}$.

Remark 1. The varieties $\overline{X}$ and $\hat{X}$ are possibly not smooth and hence the closed subsets $\overline{F}$ and $\hat{F}$ may contain singularities.

Let x be an indeterminacy point of f and a a value. The function $\overline{f}$ is well defined at (x,a) and $\overline{X}$ could be singular. Nevertheless, Lê Dung Trang in [7] defined a Milnor fiber of $\overline{f}$ at (x,a) which is topologically identical to the Milnor fiber $F_{x,a}$ considered by Gusein-Zade, Luengo and Melle Hernández in [5] and [6]. Let $\mathcal{F}_a^{\circlearrowleft}$ be the complex of constructible sheaves in $\mathcal{D}_c^b(\overline{f}^{-1}(a))^{\mathrm{mon}}$ defined by $\psi_{\overline{f}-a}(R\bar{i}_!\mathbb{Q}_X)$ (where $\overline{f}-a$ is $1/\overline{f}$ if a is infinite). The complex of sheaves associated to the Milnor fiber $F_{x,a}$ is the germ $\mathcal{F}_{x,a}^{\circlearrowleft}$ of $\mathcal{F}_a^{\circlearrowleft}$ at (x,a) naturally endowed with a monodromy action induced by the construction of the nearby cycles sheaf. As in the polynomial case this sheaf construction is connected to the topological point of view:

Proposition 2.1. *For all k there exists an isomorphism between the cohomology groups*

$$\mathcal{H}^k(\mathcal{F}_{x,a})^{\circlearrowleft} \simeq H_c^k(F_{x,a},\mathbb{Q})^{\circlearrowleft T_{x,a}}$$

endowed with their monodromy action.

This construction exists at the mixed Hodge modules level of M. Saito [11]. Thus we obtain a complex of mixed Hodge modules $\mathcal{F}_{x,a}^{\mathrm{MHM}\circlearrowleft}$ over $\{(x,a)\}$. The underlying mixed Hodge structure of $\mathcal{H}^k\mathcal{F}_{x,a}^{\mathrm{MHM}\circlearrowleft}$ is by Proposition 2.1 the *limit mixed Hodge structure* of $H_c^k(F_{x,a},\mathbb{Q})^{\circlearrowleft T_{x,a}}$.

We consider now the complex of constructible sheaves $\mathcal{G}_a^{\circlearrowleft}$ in $\mathcal{D}_c^b(\{a\})^{\mathrm{mon}}$ defined by $R\hat{f}_!\psi_{\hat{f}-a}(Ri_!\mathbb{Q}_X)$. As in the polynomial case this sheaf construction is associated to the "global" Milnor fiber F_a:

Proposition 2.2. *The complex of constructible sheaves $\mathcal{G}_a^{\circlearrowleft}$ does not depend on the compactification $(\hat{X},\hat{i},\hat{f})$ and for all k, there exists an isomorphism between $\mathcal{H}^k\mathcal{G}_a^{\circlearrowleft}$ and $H_c^k(F_a,\mathbb{Q})^{\circlearrowleft T_a}$.*

The previous construction exists at the level of mixed Hodge modules, we obtain in this way a complex of mixed Hodge modules $\mathcal{G}_a^{\mathrm{MHM}\circlearrowleft}$ over $\{a\}$. The underlying mixed Hodge structure of the mixed Hodge module $\mathcal{H}^k\mathcal{G}_a^{\mathrm{MHM}\circlearrowleft}$ is by Proposition 2.2 the *limit mixed Hodge structure* of $H_c^k(F_a,\mathbb{Q})^{\circlearrowleft T_a}$.

3. Motivic point of view

Concerning the usual definitions of Grothendieck rings, arc spaces and the motivic measure introduced by Kontsevich we refer to [1]. Let S be a fixed variety endowed with the trivial action of $\mathbb{G}_m$. We will work in the Grothendieck rings $\mathcal{M}_{S\times\mathbb{G}_m}^{\mathbb{G}_m}$ of varieties $(p_S, p_{\mathbb{G}_m}: V \to S\times\mathbb{G}_m, \sigma)$ where σ is a good action of the multiplicative group $\mathbb{G}_m$ on V, p_S is a morphism with $\mathbb{G}_m$-invariant fibers and $p_{\mathbb{G}_m}$ is a homogeneous map according to the action σ [4, § 2.2]. As in [9, Section 4] for two positive integers n and δ, we define

$$\hat{X}_n^{\delta} := \left\{\varphi \in \mathcal{L}(\hat{X}) \;\middle|\; \mathrm{ord}_t\, \varphi^*\hat{F} \leqslant n\delta, \mathrm{ord}_t\left[\hat{f}\left(\varphi(t)\right) - \hat{f}\left(\varphi(0)\right)\right] = n\right\}.$$

This semi-algebraic part of the arc space $\mathcal{L}(\hat{X})$ of $\hat{X}$ is endowed with the usual action $\lambda.\varphi(t) \mapsto \varphi(\lambda t)$ of $\mathbb{G}_m$, and the morphism $\varphi \mapsto [\varphi(0), \mathrm{ac}(\hat{f}(\varphi)(t) - \hat{f}(\varphi)(0))]$ in $\hat{X}\times\mathbb{G}_m$.

We consider the global motivic zeta function $Z_{\hat{f},X}^{\delta}(T)$ defined as $\sum_{n\geqslant 1}\mu(\hat{X}_n^{\delta})T^n$ in $\mathcal{M}_{\hat{X}\times\mathbb{G}_m}^{\mathbb{G}_m}[[T]]$.

For the following results and definitions see [2, § 3.5], [4, § 3.8] and [9, § 4.4]:

Theorem 3.1. *There exists δ_0 such that for all $\delta > \delta_0$, the motivic zeta function $Z^{\delta}_{\hat{f},X}(T)$ is rational and its limit exists as T goes to ∞. The limit $-\lim_{T\to\infty} Z^{\delta}_{\hat{f},X}(T)$, denoted by $S^{\mathrm{global}}_{\hat{f},X}$, is in $\mathcal{M}^{\mathbb{G}_m}_{\hat{X}\times\mathbb{G}_m}$.*

Definition 3.2. The *motivic Milnor fiber of f for the value a at an indeterminacy point x* is defined as

$$S_{f,x,a} := i^{-1}_{\{(x,a)\}\times\mathbb{G}_m} S^{\mathrm{global}}_{\hat{f},X} \in \mathcal{M}^{\mathbb{G}_m}_{\{(x,a)\}\times\mathbb{G}_m},$$

where $i^{-1}_{\{(x,a)\}\times\mathbb{G}_m}$ is the restriction morphism from $\mathcal{M}^{\mathbb{G}_m}_{\hat{X}\times\mathbb{G}_m}$ to $\mathcal{M}^{\mathbb{G}_m}_{\{(x,a)\}\times\mathbb{G}_m}$ see [4, § 2.3].

Theorem 3.3. *The motivic Milnor fiber $S_{f,x,a}$ of f for the value a at an indeterminacy point x does not depend on $(\hat{X}, \hat{i}, \hat{f})$ and can be computed via arcs φ with $\varphi(0) = x$ and $\hat{f}(\varphi(0)) = a$.*

Definition 3.4. A value a is *motivically atypical* for f at x if the motive $S_{f,x,a}$ is different from zero. This set of motivically atypical values denoted by $B^{mot}_{f,x}$ is called the *motivic bifurcation set of f at x*.

Lemma 3.5. *Let (Z, E, h) be a log-resolution of $(\hat{X}, \hat{F} \cup \hat{f}^{-1}(\mathrm{disc}(f)))$ and $(E_i)_{i\in A}$ be the irreducible components of E. For all $I \subset A$ denote $\bigcap_{i\in I} E_i \setminus \bigcup_{j\notin I} E_j$ by E^0_I where $E^0_\emptyset$ is $Z \setminus E$. There is a decomposition*

$$S^{\mathrm{global}}_{\hat{f},X} = \big[(\bar{i}, pr_{\mathbb{G}_m}) : X \setminus \hat{f}^{-1}\big(\mathrm{disc}(\hat{f}, Z)\big) \times \mathbb{G}_m \to \hat{X} \times \mathbb{G}_m\big] + \sum_{a\in \mathrm{disc}(\hat{f},Z)} S_{\hat{f},a,X} \in \mathcal{M}^{\mathbb{G}_m}_{\hat{X}\times\mathbb{G}_m}$$

where the discriminant $\mathrm{disc}(\hat{f}, Z)$ is the finite set $\bigcup_{I\subset A} \mathrm{disc}(\hat{f} \circ h)_{|E^0_I}$ and for all a, $S_{\hat{f},a,X}$ is the restriction of $S^{\mathrm{global}}_{\hat{f},X}$ to $\hat{f}^{-1}(a) \times \mathbb{P}^1_{\mathbb{C}}$. Then in $\mathcal{M}^{\mathbb{G}_m}_{\hat{X}\times\mathbb{G}_m}$, using the motivic vanishing cycles we have the equality

$$S^{\mathrm{global},\phi}_{\hat{f},X} = \sum_{a\in\mathrm{disc}(\hat{f},Z)} S^{\phi}_{\hat{f},a,X},$$

where $S^{\mathrm{global},\phi}_{\hat{f},X}$ is $(-1)^{d-1}(S^{\mathrm{global}}_{\hat{f},X} - [X \times \mathbb{G}_m])$ and $S^{\phi}_{\hat{f},a,X}$ is $(-1)^{d-1}(S_{\hat{f},a,X} - [(f^{-1}(a) \times \{a\}) \times \mathbb{G}_m])$.

Theorem 3.6. *The set $B^{mot}_{f,x}$ is finite by Lemma 3.5 and in $\mathcal{M}^{\mathbb{G}_m}_{(\{x\}\times\mathbb{P}^1_{\mathbb{C}})\times\mathbb{G}_m}$ we have the equality*

$$S_{f,x} = \sum_{a\in B^{mot}_{f,x}} S_{f,x,a}$$

where $S_{f,x}$ is $i^{-1}_{(\{x\}\times\mathbb{P}^1_{\mathbb{C}})\times\mathbb{G}_m} S^{\mathrm{global}}_{\hat{f},X}$.

Let (Y, E, h) be a log-resolution of $(\hat{X}, (\{P = 0\} \cup \{Q = 0\}) \times \mathbb{P}^1_{\mathbb{C}})$ such that $h^{-1}(x, a)$ is a normal crossing divisor. Let E_i be the irreducible components of E indexed by A, and for all nonempty subset I of A, let E^0_I be $\bigcap_{i\in I} E_i \setminus \bigcup_{i\notin I} E_i$. Let U_I be the fibred product, over E^0_I, of the normal bundles of E_i without zero-section, let σ be the natural diagonal action of $\mathbb{G}_m$ on U_I and f_I the quotient of monomial residue functions P_I, Q_I induced by P, Q [4, § 3.4]. Let $A(x, a)$ be the index-set of the irreducible components of $h^{-1}(x, a)$. We denote by $(N_i(P))$ and $(N_i(Q))$ the multiplicities of the divisors $\mathrm{div}\, P \circ h$ and $\mathrm{div}\, Q \circ h$ along the irreducible components E_i. We define $\mathcal{A}$ as $\{I \subset A(x, a) \mid I \neq \emptyset$ and $\forall i \in I, N_i(P) > N_i(Q)\}$. For all I in $\mathcal{A}$, the bundle U_I is over $\{(x, a)\}$ and we denote by p_I the structural map.

Proposition 3.7. *The motivic Milnor fiber of the germ of f at x for the value a is*

$$S_{f,x,a} = -\sum_{I\in\mathcal{A}} (-1)^{|I|} \big[(p_I, f_I) : U_I \to \big\{(x, a)\big\} \times \mathbb{G}_m, \sigma\big] \in \mathcal{M}^{\mathbb{G}_m}_{\{(x,a)\}\times\mathbb{G}_m}.$$

Definition 3.8. We denote by $\mathcal{M}^{\hat{\mu}}_{\mathrm{Spec}\,\mathbb{C}}$ the Grothendieck ring of varieties over the point endowed with a good action of the group of roots of unity $\hat{\mu}$ and by $K_0(\mathrm{MHM}^{\mathrm{mon}}_{\mathrm{Spec}\,\mathbb{C}})$ the Grothendieck ring of the mixed Hodge modules supported by the point and endowed with a quasi-unipotent operator. The Hodge realization

$$\chi^{\mathrm{MHM}} : \mathcal{M}^{\hat{\mu}}_{\mathrm{Spec}\,\mathbb{C}} \to K_0\big(\mathrm{MHM}^{\mathrm{mon}}_{\mathrm{Spec}\,\mathbb{C}}\big),$$

associates to the class $[A, \sigma]$ with A a smooth and proper variety and σ a good action, the class $[Q^{\mathrm{MHM}}_A, \sigma]$, where Q^{MHM}_A is the trivial Hodge module with the endowed action [4, § 3.16].

As a corollary of [1, § 4.2] and [4, § 3.7], by comparison of objects via such log-resolution we obtain:

Theorem 3.9. *The fiber over* 1 *of* $S_{f,x,a}$ *realizes on the class of the mixed Hodge module* $[\mathcal{F}_{x,a}^{\mathrm{MHM}\circlearrowleft}]$

$$\chi^{\mathrm{MHM}}\big(S_{f,x,a}^{(1)}\big) = \big[\mathcal{F}_{x,a}^{\mathrm{MHM}\circlearrowleft}\big] \in K_0\big(\mathrm{MHM}_{x,a}^{\mathrm{mon}}\big).$$

Remark 2. By Proposition 3.7, we obtain an expression of the class of the limit mixed Hodge structure and its spectrum.

Definition 3.10. For any integer n, the *nth-Lefschetz number of the monodromy* of f at the point x for the value a is $\sum_k (-1)^k \operatorname{trace}(T_{x,a}^n, H_c^k(F_{x,a}, \mathbb{Q}))$, usually denoted by $\Lambda^n(T_{x,a})$.

As in the polynomial case [3], using the formula of the monodromy zeta function in [5] we have:

Theorem 3.11. *There exists* δ_0 *such that for all* $\delta > \delta_0$ *and for all integer n,*

$$Eu_c\big(\mu\big(\hat{X}_{n,x,a}^{\delta,1}\big)\big) = \Lambda^n(T_{x,a}) = \sum_{i \in A_n} \big(N_i(P) - N_i(Q)\big) Eu_c\big(E_i^0\big)$$

with $A_n := \{i \in A(x,a) \mid N_i(P) > N_i(Q),\ N_i(P) - N_i(Q) \text{ divides } n\}$ *and* $\hat{X}_{n,x,a}^{\delta,1}$ *is the fiber of* $\hat{X}_n^{\delta}$ *over* $(x, a, 1)$. *In particular by the above log-resolution we can take* δ_0 *equal to* $\max(N_i(P)/(N_i(P) - N_i(Q)))$.

Now we define for a value a, a motivic analogue $S_{f,a}$ for the "global" Milnor fiber F_a and the sheaf $\mathcal{G}_a$.

Definition 3.12. We define S_f^{global} as $p_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{G}_m !} S_{\hat{f},X}^{\mathrm{global}}$ and $S_f^{\mathrm{global},\phi}$ as $(-1)^{d-1}(S_f^{\mathrm{global}} - [X \times \mathbb{G}_m])$ in $\mathcal{M}_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{G}_m}^{\mathbb{G}_m}$ where $p_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{G}_m !}$ is the direct image morphism from $\mathcal{M}_{\hat{X} \times \mathbb{G}_m}^{\mathbb{G}_m}$ to $\mathcal{M}_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{G}_m}^{\mathbb{G}_m}$ induced by the composition with $(\hat{f}, id_{\mathbb{G}_m})$ [4, § 2.3].

Definition 3.13. The *motivic Milnor fiber of* f *for a value* a, denoted by $S_{f,a}$, is $i_{\{a\} \times \mathbb{G}_m}^{-1} S_f^{\mathrm{global}}$ in $\mathcal{M}_{\{a\} \times \mathbb{G}_m}^{\mathbb{G}_m}$, and its motivic vanishing cycles $S_{f,a}^{\phi}$ are defined as $(-1)^{d-1}(S_{f,a} - [(f^{-1}(a) \times \{a\}) \times \mathbb{G}_m])$.

Analogous to the local case we have the following realization theorem as in [8] and [9]:

Theorem 3.14. *The motives* S_f^{global}, $S_{f,a}$ *and* $S_{f,a}^{\phi}$ *do not depend on the compactification* $\hat{X}$ *and*

$$\chi^{\mathrm{MHM}}\big(S_{f,a}^{(1)}\big) = \big[\mathcal{G}_a^{\mathrm{MHM}\circlearrowleft}\big] \in K_0\big(\mathrm{MHM}_a^{\mathrm{mon}}\big).$$

Definition 3.15. A value a is *motivically atypical* for f if it is a critical value or if the motivic vanishing cycles $S_{f,a}^{\phi}$ are different from zero. These values constitute the *motivic bifurcation set* B_f^{mot} *of* f.

Theorem 3.16. *The motivic bifurcation set is finite by Lemma* 3.5 *and we have the equality in* $\mathcal{M}_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{G}_m}^{\mathbb{G}_m}$

$$S_f^{\mathrm{global},\phi} = \sum_{a \in B_f^{mot}} S_{f,a}^{\phi}.$$

Acknowledgements

I thank Alejandro Melle Hernández who led me to this problem, Claude Sabbah for explaining to me the sheaf construction in Section 2 and François Loeser for his guidance. This project has been supported by the Instituto de Matemática Interdisciplinar, by the ERC Advanced Grant NMNAG and the project ANR-08-JCJC-0118-01.

References

[1] Jan Denef, François Loeser, Motivic Igusa zeta functions, J. Algebraic Geom. 7 (3) (1998) 505–537.
[2] Jan Denef, François Loeser, Geometry on arc spaces of algebraic varieties, in: European Congress of Mathematics, Vol. I, Barcelona, 2000, in: Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 327–348.
[3] Jan Denef, François Loeser, Lefschetz numbers of iterates of the monodromy and truncated arcs, Topology 41 (5) (2002) 1031–1040.
[4] Gil Guibert, François Loeser, Michel Merle, Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink, Duke Math. J. 132 (3) (2006) 409–457.

[5] Sabir Gusein-Zade, Ignacio Luengo, Alejandro Melle Hernández, Zeta functions of germs of meromorphic functions, and the Newton diagram, Funktsional. Anal. i Prilozhen. 32 (2) (1998) 26–35, 95.
[6] Sabir Gusein-Zade, Ignacio Luengo, Alejandro Melle Hernández, Bifurcations and topology of meromorphic germs, in: New Developments in Singularity Theory, Cambridge, 2000, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 21, Kluwer Acad. Publ., Dordrecht, 2001, pp. 279–304.
[7] Dung Trang Le, Some remarks on relative monodromy, in: Real and Complex Singularities, Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 397–403.
[8] Yutaka Matsui, Kiyoshi Takeuchi, Monodromy at infinity of polynomial maps and mixed Hodge modules, arXiv:0912.5144v5.
[9] Michel Raibaut, Singularités à l'infini et intégration motivique, Bull. Soc. Math. France 140 (1) (2012) 51–100.
[10] Michel Raibaut, Motivic Milnor fibers of a rational function, submitted for publication.
[11] Morihiko Saito, Mixed Hodge modules, Publ. Res. Inst. Math. Sci. 26 (2) (1990) 221–333.