C. R. Acad. Sci. Paris, Ser. I 350 (2012) 413–416

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential Geometry

Lie geometry of linear Weingarten surfaces

La géométrie de Lie des surfaces de Weingarten linéaires

Francis E. Burstall [a], Udo Hertrich-Jeromin [a], Wayne Rossman [b]

[a] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK*
[b] *Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan*

A R T I C L E I N F O

Article history:
Received 18 January 2011
Accepted 23 March 2012
Available online 31 March 2012

Presented by Jean-Pierre Demailly

A B S T R A C T

We show how linear Weingarten surfaces appear as special Ω-surfaces and give a characterization of those linear Weingarten surfaces that allow a Weierstrass type representation.

R É S U M É

Nous montrons que les surfaces de Weingarten linéaires peuvent être présentées comme des surfaces Ω spéciales. Ensuite, nous discutons une caractérisation des surfaces de Weingarten linéaires de type Bryant.

1. Introduction

We demonstrate how (spacelike) linear Weingarten surfaces in (Lorentzian) space forms appear as Ω- or Ω_0-surfaces with a (pair of) isothermic sphere congruence(s) that each take values in a linear sphere complex. One virtue of this Lie geometric approach is that the rich isothermic transformation theory becomes available. For example, a "Lawson correspondence" for linear Weingarten surfaces is immediately obtained from the Lie geometric deformation of Ω-surfaces [7]: the fixed sphere complexes are constant conserved quantities of the enveloped isothermic sphere congruences, that is, they are fixed by the respective Calapso transformations which provide the Lie geometric deformation of the Legendre lift of the surface; consequently, this characterizing property is preserved by the deformation.

In the special case where one of the enveloped isothermic sphere congruences envelops a fixed sphere, hence giving rise to a holomorphic map into a Riemann sphere, those linear Weingarten surfaces that allow a Weierstrass type representation are obtained. This observation leads to simple proofs of recent results about Bryant type surfaces in hyperbolic space, cf. [6].

Besides the implications for smooth linear Weingarten surfaces, our approach also lends itself to discretization: a canonical Lie geometric definition of discrete linear Weingarten surfaces as special Ω-nets will lead to a similar theory in the discrete case.

2. Linear Weingarten surfaces in space forms

Fix a point sphere complex $\mathfrak{p} \in \mathbb{R}^{4,2}$, $|\mathfrak{p}|^2 \neq 0$, and a space form vector $\mathfrak{q} \perp \mathfrak{p}$ to consider a surface

$$\mathfrak{f} : M^2 \to \mathfrak{Q}^3 := \{ y \in \mathcal{L}^5 \mid (y, \mathfrak{p}) = 0,\ (y, \mathfrak{q}) = -1 \}$$

E-mail addresses: feb@maths.bath.ac.uk (F.E. Burstall), u.hertrich-jeromin@bath.ac.uk (U. Hertrich-Jeromin), wayne@math.kobe-u.ac.jp (W. Rossman).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.03.018

in the quadric $\mathfrak{Q}^3$ of constant curvature $\kappa = -|\mathfrak{q}|^2$, where $\mathcal{L}^5 \subset \mathbb{R}^{4,2}$ denotes the light cone; its tangent plane congruence

$$\mathfrak{t} : M^2 \to \mathfrak{P}^3 := \{ y \in \mathcal{L}^5 \mid (y, \mathfrak{p}) = -1,\ (y, \mathfrak{q}) = 0 \} \quad \text{with } (\mathfrak{t}, \mathfrak{f}) = 0 \text{ and } (\mathfrak{t}, d\mathfrak{f}) \equiv 0.$$

Away from umbilics we introduce curvature line coordinates (u, v), so that Rodrigues' equations hold: $0 = \mathfrak{t}_u + \kappa_1 \mathfrak{f}_u = \mathfrak{t}_v + \kappa_2 \mathfrak{f}_v$. Now $\mathfrak{f}$ is a linear Weingarten surface if there is a non-trivial (real) linear combination

$$0 = aK + 2bH + c, \quad \text{where } H = \frac{\kappa_1 + \kappa_2}{2} \text{ and } K = \kappa_1 \kappa_2$$

are the mean and (extrinsic) Gauss curvatures[1] of $\mathfrak{f}$. Expressing the principal curvatures $\kappa_i = \frac{(s_i, \mathfrak{q})}{(s_i, \mathfrak{p})}$ in terms of the curvature spheres $s_i = \mathfrak{t} + \kappa_i \mathfrak{f}$, the linear Weingarten condition reads

$$0 = k_1 W k_2^t \quad \text{with } k_i := \big((s_i, \mathfrak{q}), (s_i, \mathfrak{p}) \big) \text{ and } W := \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Evidently, a change of basis of the plane $\langle \mathfrak{q}, \mathfrak{p} \rangle$ does not change the shape of the linear Weingarten condition: if $(\tilde{\mathfrak{q}}, \tilde{\mathfrak{p}}) = (\mathfrak{q}, \mathfrak{p}) B^{-1}$, $B \in Gl(2)$, denotes another basis[2] then

$$\tilde{k}_i = k_i B^{-1} \quad \text{and} \quad 0 = \tilde{k}_1 \tilde{W} \tilde{k}_2^t \quad \text{with } \tilde{W} = B W B^t.$$

In particular, the parallel surfaces of a linear Weingarten surface in a space form (choosing B so that inner products are preserved) are linear Weingarten where they immerse. Also, this observation provides a duality for linear Weingarten surfaces in H^3 and in $S^{2,1}$, cf. [6]: when $|\mathfrak{p}|^2 = -1$ and $|\mathfrak{q}|^2 = 1$ and the surface $\mathfrak{f} : M^2 \to \mathfrak{Q}^3 \cong H^3$ has regular Gauss map $\mathfrak{t} : M^2 \to \mathfrak{P}^3 \cong S^{2,1}$ then, swapping the roles of the point sphere complex and the space form vector, $\mathfrak{t}$ becomes a spacelike linear Weingarten surface with mean and Gauss curvatures $\frac{H}{K}$ and $\frac{1}{K}$, respectively.

Linear Weingarten surfaces in space forms come in various flavors. If $a = 0$ then $\mathfrak{f}$ is a surface of constant mean curvature, hence, as an isothermic surface, an Ω-surface with its central sphere congruence as the second enveloped isothermic sphere congruence, see [3]. If $a \neq 0$ then the linear Weingarten condition can be rewritten as $(a\kappa_1 + b)(a\kappa_2 + b) + (ac - b^2) = 0$; if now the discriminant $ac - b^2 \neq 0$ then, depending on the sign of the discriminant, there is a real function φ so that

$$\left.\begin{aligned} a\kappa_1 + b &= \sqrt{b^2 - ac}\,\tanh\varphi, \\ a\kappa_2 + b &= \sqrt{b^2 - ac}\,\coth\varphi; \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a\kappa_1 + b &= \sqrt{ac - b^2}\,\tan\varphi, \\ a\kappa_2 + b &= -\sqrt{ac - b^2}\,\cot\varphi \end{aligned}\right.$$

hence, by the Codazzi equations, curvature line parameters can be adjusted so that

$$E = \cosh^2\varphi, \qquad G = \sinh^2\varphi \quad \text{or} \quad E = \cos^2\varphi, \qquad G = \sin^2\varphi.$$

Thus, with $C = \frac{a^2}{ac - b^2}$, Calapso's equation [2] $CEG(\kappa_1 - \kappa_2)^2 = G \mp E$ holds, showing that these linear Weingarten surfaces are Guichard surfaces, hence Ω-surfaces, see [3]. In the second case, the pair of enveloped isothermic sphere congruences becomes complex conjugate. Finally, if the discriminant $ac - b^2 = 0$ then one of the principal curvatures is constant so that $\mathfrak{f}$ becomes a tube[3] over a space curve. In this case the two isothermic sphere congruences coincide with the enveloped 1-parameter family of curvature spheres and the surface is an Ω_0-surface.

3. Linear Weingarten surfaces as special Ω-surfaces

Using the aforementioned freedom of choice of basis for the plane $\langle \mathfrak{q}, \mathfrak{p} \rangle$ we shall give a unified, Lie geometric analysis for the non-degenerate cases: by Sylvester's inertia theorem, we may choose a basis (q_1, q_2) so that the linear Weingarten condition reads, with $\varepsilon \in \{0, 1, i\}$,

$$0 = (s_1, q_1)(s_2, q_1) - \varepsilon^2 (s_1, q_2)(s_2, q_2).$$

If $\varepsilon \neq 0$, that is, $ac - b^2 \neq 0$, we may rescale the curvature spheres so that, with a suitable function φ, $(s_1, q_2) = (s_2, q_1) = \cosh\varepsilon\varphi$ and $(s_1, q_1) = \varepsilon^2 (s_2, q_2) = \frac{1}{\varepsilon}\sinh\varepsilon\varphi$. As the s_i are curvature spheres, that is, $s_{1u}, s_{2v} \in \langle s_1, s_2 \rangle$, we infer that $s_{1u} = \varphi_u s_2$ and $s_{2v} = \varepsilon^2 \varphi_v s_1$. Hence $s^{\pm}_{uv} = \varepsilon^2 (\varphi_u \varphi_v \pm \varepsilon \varphi_{uv}) s^{\pm}$, where $s^{\pm} := s_1 \pm \varepsilon s_2$, so that $s^{\pm}$ define a (if $\varepsilon = i$ complex conjugate) pair of enveloped isothermic sphere congruences for the Legendre immersion $f = \langle \mathfrak{f}, \mathfrak{t} \rangle$, which is therefore an Ω-surface, see [3] and [4]. Moreover, the isothermic sphere congruences each take values in a fixed linear sphere complex[4] as $(s^{\pm}, q^{\pm}) \equiv 0$ for $q^{\pm} := q_1 \mp \varepsilon q_2$.

---

[1] In contrast to some authors we do not equip the Gauss curvature with a sign in the case $|\mathfrak{p}|^2 > 0$ of a spacelike surface in a Lorentzian space form. Our tangent plane congruence $\mathfrak{t}$ yields a constant length normal field in the space form; the normalization $|\mathfrak{p}|^2 = \mp 1$ yields the usual principal curvatures κ_i, i.e., with respect to a normalized Gauss map.

[2] Note that, as we exclude umbilics, $\tilde{k}_1$ and $\tilde{k}_2$ are linearly independent.

[3] Possibly with infinite radius.

[4] That is, $q^{\pm}$ are constant conserved quantities for $s^{\pm}$: $(d + \lambda\tau^{\pm}) q^{\pm} = 0$ for all $\lambda \in \mathbb{R}$, where $d + \lambda\tau^{\pm}$ defines the isothermic loop of (flat) connections of $s^{\pm}$, cf. [1].

Conversely, suppose that $f = \langle s^+, s^- \rangle$ is an Ω-surface, given in terms of a (possibly complex conjugate) pair of isothermic sphere congruences $s^\pm$, each of which takes values in a linear sphere complex[5] $q^\pm$. As the $s^\pm$ separate the curvature spheres s_i of f harmonically, we may assume that their Moutard lifts are aligned to reflect across the Lie-cyclides of f, that is, for suitable lifts of the curvature spheres[6]

$$s^\pm = s_1 \pm \varepsilon s_2, \quad \text{hence} \quad (s_1, q^+)(s_2, q^-) + (s_1, q^-)(s_2, q^+) = 0,$$

showing that f projects to a non-tubular linear Weingarten surface, $ac - b^2 \neq 0$, in any space form given by a choice of point sphere complex and space form vector $\mathfrak{p}, \mathfrak{q} \in \langle q^+, q^- \rangle$.

Thus we have proved: *Linear Weingarten surfaces with $ac - b^2 \neq 0$ in space forms are those Ω-surfaces $f = \langle s^+, s^- \rangle$ enveloping a (possibly complex conjugate) pair of isothermic sphere congruence $s^\pm$, each of which takes values in a linear sphere complex $q^\pm$. The plane spanned by $q^\pm$ is the plane of the point sphere complex*[7] *$\mathfrak{p}$ and the space form vector $\mathfrak{q}$.*

It remains to investigate the degenerate case $\varepsilon = 0$, that is, $ac - b^2 = 0$, of a tube over a curve: in this case we may assume $(s_2, q_1) \equiv 0$ and $(s_1, q_1) \neq 0$; hence $s := s_2$ can be normalized so that $s_v \equiv 0$, showing that s is an isothermic sphere congruence that takes values in the linear sphere complex q_1. Conversely, if $f = \langle s_1, s_2 \rangle$ is a Legendre immersion so that the curvature sphere congruence s_2 is isothermic, $s_{2uv} \parallel s_2$, and maps into a fixed linear sphere complex q then, for any choice of a complementary linear sphere complex q_2,

$$(s_1, q)(s_2, q) - 0(s_1, q_2)(s_2, q_2) = 0,$$

showing that f projects to a tube in a suitable space form. *Tubes over curves in space forms, i.e., linear Weingarten surfaces with $ac - b^2 = 0$, are those Ω_0-surfaces whose isothermic curvature sphere congruence takes values in a linear sphere complex.*

In contrast to the non-degenerate case, the space form projection is far more flexible in the degenerate case: for example, if $|q|^2 < 0$ then $\mathfrak{p} := q$ is the point sphere complex for a (definite) conformal subgeometry of Lie geometry, where f projects to the curve $\mathfrak{f} \simeq s_2$; then choosing a space form vector $\mathfrak{q} \perp \mathfrak{p}$ specifies a space form subgeometry of this conformal geometry and subsequent parallel transformations in this space form yield tubes around the original curve.

4. Weierstrass representations

If one of the sphere complexes $q^\pm$ obtained from a (non-degenerate) linear Weingarten surface, say q^+, is given by a (real) sphere, $|q^+|^2 = 0$, then the second envelope of s^+ yields a holomorphic map.[8] We shall see that suitable space form projections $\mathfrak{f} = -\frac{1}{\Delta}(s^+ \wedge s^-)\mathfrak{p}$ and $\mathfrak{t} = \frac{1}{\Delta}(s^+ \wedge s^-)\mathfrak{q}$ with $s^\pm = s_1 \pm s_2$ and $\Delta := ((s^+ \wedge s^-)\mathfrak{p}, \mathfrak{q}) = (s^+, \mathfrak{p})(s^-, \mathfrak{q}) - (s^-, \mathfrak{p})(s^+, \mathfrak{q}) \neq 0$ yield (parallel surfaces of) surfaces that are known to allow a Weierstrass representation.

In the case $q^- \perp q^+$ of a degenerate metric on $\langle q^+, q^- \rangle$ we choose[9] $\mathfrak{q} := q^+$ and $\mathfrak{p} := q^-$ to find

$$k_1 = \big((s_1, \mathfrak{q}), (s_1, \mathfrak{p})\big) = \frac{1}{2}\big((s^-, q^+), (s^+, q^-)\big) \quad \text{and} \quad k_2 = \frac{1}{2}\big(-(s^-, q^+), (s^+, q^-)\big).$$

Hence $k_1 W k_2^t = 0$ for $W = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, showing that $\mathfrak{f}$ is a minimal surface in either Euclidean space or in Minkowski space, depending on the sign of $|\mathfrak{p}|^2$; in both cases a Weierstrass representation is known, see [5].

In the non-degenerate case $(q^+, q^-) \neq 0$ we may assume that $(q^+, q^-) = -\frac{1}{2}$ and set $\mu := |q^-|^2$. Then $\mathfrak{q} := q^- + (\mu - 1)q^+$ and $\mathfrak{p} := q^- + (\mu + 1)q^+$ define a space form projection $\mathfrak{f} : M^2 \to H^3$ into hyperbolic space with

$$k_1 = \frac{1}{2}\big((s^+, q^-) + (\mu - 1)(s^-, q^+), (s^+, q^-) + (\mu + 1)(s^-, q^+)\big),$$
$$k_2 = \frac{1}{2}\big((s^+, q^-) - (\mu - 1)(s^-, q^+), (s^+, q^-) - (\mu + 1)(s^-, q^+)\big);$$

hence $k_1 W k_2^t = 0$ with $W = \left(\begin{smallmatrix} \mu+1 & -\mu \\ -\mu & \mu-1 \end{smallmatrix}\right)$, showing that $\mathfrak{f}$ is a linear Weingarten surface of Bryant type, cf. [6]. These are the linear Weingarten surfaces in hyperbolic space known to allow a Weierstrass representation; the flat front case discussed in [1] is obtained at $\mu = 0$. *The non-degenerate linear Weingarten surfaces one of whose enveloped isothermic sphere congruences envelops a fixed sphere allow a Weierstrass type representation.*

In [6] the authors observe that non-degenerate linear Weingarten surfaces of Bryant type come in three types: flat fronts and surfaces that are parallel to either a cmc-1 surface ($\mu = -1$) or an hmc-1 surface ($\mu = 1$). In our setup this is reflected

[5] The $q^\pm$ are linearly independent: the assumption $q^+ = q^-$ would lead to f being totally umbilic.

[6] In the complex conjugate case we may, without loss of generality, assume that $q^\pm = q_1 \mp iq_2$ are complex conjugate as well: the linear Weingarten condition then becomes real in terms of q_1 and q_2.

[7] In the case of a spacelike point sphere complex, $|\mathfrak{p}|^2 > 0$, a spacelike linear Weingarten surface in a Lorentzian space form is obtained. We need to exclude the case of $\langle q^+, q^- \rangle$ being a null 2-plane though.

[8] Holomorphic with respect to the conformal structure induced by the isothermic sphere congruences $s^\pm$: for the Bryant type surfaces below this is given by $\mathrm{I} - 2\mathrm{II} + \mathrm{III}$ or, equivalently, by $(\mu - 1)^2\mathrm{I} - 2(\mu - 1)(\mu + 1)\mathrm{II} + (\mu + 1)^2\mathrm{III}$.

[9] We need to exclude the case of a totally degenerate metric, where $\langle q^+, q^- \rangle$ is a null 2-plane; using q^+ to project to Laguerre geometry will shed light on this case as well, cf. [8].

by the remaining scaling freedom of q^-, a rescaling providing a parallel transformation in H^3. The aforementioned duality of linear Weingarten surfaces in H^3 and $S^{2,1}$ exchanges cmc-1 surfaces and hmc-1 surfaces while flatness of the induced metric is preserved.

Acknowledgements

We would like to thank D. Calderbank and M. Umehara for fruitful discussions.

References

[1] F. Burstall, U. Hertrich-Jeromin, W. Rossman, Lie geometry of flat fronts in hyperbolic space, C. R. Acad. Sci. Paris, Sér. I 348 (2010) 661–664.
[2] P. Calapso, Alcune superficie di Guichard e le relative trasformazioni, Ann. Mat. 11 (1904) 201–251.
[3] A. Demoulin, Sur les surfaces R et les surfaces Ω, C. R. Acad. Sci. Paris 153 (1911) 590–593, 705–707.
[4] A. Demoulin, Sur les surfaces Ω, C. R. Acad. Sci. Paris 153 (1911) 927–929.
[5] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space L^3, Tokyo J. Math. 6 (1983) 297–309.
[6] M. Kokubu, M. Umehara, Orientability of linear Weingarten surfaces, spacelike cmc-1 surfaces and maximal surfaces, Math. Nachr. 284 (2011) 1903–1918.
[7] E. Musso, L. Nicolodi, Deformation and applicability of surfaces in Lie sphere geometry, Tôhoku Math. J. 58 (2006) 161–187.
[8] A. Szereszewski, L-isothermic and L-minimal surfaces, J. Phys. A: Math. Theor. 42 (2009) 115203.