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On the regular convergence of multiple integrals of locally Lebesgue integrable functions over $\overline{\mathbb{R}}^m_+$

Sur la convergence régulière d'intégrales multiples définies sur $\overline{\mathbb{R}}^m_+$ localement intégrables au sens de Lebesgue

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A R T I C L E   I N F O

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A B S T R A C T

Let the function $f : \overline{\mathbb{R}}^m_+ \to \mathbb{C}$ be such that $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^m_+)$, where $m \geqslant 2$ is a fixed integer. We investigate the convergence behavior of the m-multiple integral

$$\int_0^{v_1}\int_0^{v_2}\dots\int_0^{v_m} f(t_1,t_2,\dots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\dots\mathrm{d}t_m \quad \text{as } \min\{v_1,v_2,\dots,v_m\}\to\infty, \qquad (*)$$

while using two notions of convergence: the one in Pringsheim's sense and the one in the regular sense. For the sake of brevity, we present our main result in the case $m = 2$ as follows: If $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$ and the double integral $(*)$ converges regularly, then the finite limits $\lim_{v_2\to\infty}\int_0^{v_1}(\int_0^{v_2} f(t_1,t_2)\,\mathrm{d}t_2)\,\mathrm{d}t_1 =: J_1(v_1)$ and $\lim_{v_1\to\infty}\int_0^{v_2}(\int_0^{v_1} f(t_1,t_2)\,\mathrm{d}t_1)\,\mathrm{d}t_2 =: J_2(v_2)$ exist uniformly in $0 < v_1, v_2 < \infty$, respectively, and $\lim_{v_1\to\infty} J_1(v_1) = \lim_{v_2\to\infty} J_2(v_2) = \lim_{v_1,v_2\to\infty}\int_0^{v_1}\int_0^{v_2} f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$. This can be considered as a generalized version of Fubini's theorem on successive integration in the case when $f \notin L^1(\overline{\mathbb{R}}^2_+)$.

R É S U M É

Soit $f : \overline{\mathbb{R}}^m_+ \to \mathbb{C}$ telle que $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^m_+)$, où m est un entier fixé. On étudie la convergence de l'intégrale multiple d'ordre m, $\int_0^{v_1}\int_0^{v_2}\dots\int_0^{v_m} f(t_1,t_2,\dots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\dots\mathrm{d}t_m$ quand $\min\{v_1,v_2,\dots,v_m\}\to\infty$, en utilisant deux méthodes de convergence, l'une au sens de Pringsheim, et l'autre au sens régulier. Pour simplifier on présente notre résultat fondamental pour $m = 2$, de la façon suivante : Si $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$ et si l'intégrale double converge régulièrement, alors les limites finies $\lim_{v_2\to\infty}\int_0^{v_1}(\int_0^{v_2} f(t_1,t_2)\,\mathrm{d}t_2)\,\mathrm{d}t_1 =: J_1(v_1)$ et $\lim_{v_1\to\infty}\int_0^{v_2}(\int_0^{v_1} f(t_1,t_2)\,\mathrm{d}t_1)\,\mathrm{d}t_2 =: J_2(v_2)$ existent uniformément dans $0 < v_1, v_2 < \infty$, respectivement, et on a $\lim_{v_1\to\infty} J_1(v_1) = \lim_{v_2\to\infty} J_2(v_2) = \lim_{v_1,v_2\to\infty}\int_0^{v_1}\int_0^{v_2} f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$. Ceci peut être considéré comme une généralisation du théorème de Fubini concernant l'intégration successive au cas où $f \notin L^1(\overline{\mathbb{R}}^2_+)$.

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Soit $m \geqslant 2$, un entier donné. L'intégrale d'ordre m (10) d'une fonction $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^m_+)$, où les intégrales partielles rectangulaires sont définies dans (11), est dite convergente au sens de Pringsheim vers une limite $I \in \mathbb{C}$, si pour tout $\varepsilon > 0$, il existe $\rho_1 = \rho_1(\varepsilon) \in \mathbb{R}_+$ telle que (12) soit satisfaite. De plus l'intégrale multiple d'ordre m (10) est dite régulièrement convergente si pour tout $\varepsilon > 0$ il existe $\rho_2 = \rho_2(\varepsilon) \in \mathbb{R}_+$ telle que la condition (14) soit satisfaite, où l'intégrale sous rectangulaire $I(u_1, v_1; u_2, v_2; \ldots; u_m, v_m)$ est définie dans (13). Il résulte de (14) que l'on a :

$$\left|I(v_1, v_2, \ldots, v_m) - I(u_1, u_2, \ldots, u_m)\right| \leqslant m\varepsilon, \quad \text{si } \min\{v_1, v_2, \ldots, v_m; u_1, u_2, \ldots, u_m\} > \rho_2(\varepsilon).$$

Puisque ε est quelconque, d'après le critère de convergence de Cauchy, on conclut que la convergence régulière de (10) implique la convergence au sens de Pringsheim. Par conséquent, la limite d'une intégrale multiple régulièrement convergente est bien définie. D'après la caractérisation donnée au Théorème 2.1, il est clair qu'une intégrale multiple peut converger au sens de Pringsheim sans converger régulièrement. Il est clair que si $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^m_+)$ alors l'intégrale multiple d'ordre m (10) converge régulièrement, et sa limite I est égale à l'intégrale de Lebesgue de f sur l'octant entier $\overline{\mathbb{R}}^m_+$. Le résultat inverse n'est pas vrai en général.

Notre résultat fondamental est énoncé dans le Théorème 3.1 qui peut être considéré comme une version généralisée des intégrations successives sous la condition que $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^m_+)$, et que l'inégrale multiple d'ordre m converge régulièrement. Dans le cas particulier où $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^m_+)$, le Théorème 3.1 coïncide avec le théorème classique de Fubini.

On considère des concepts analogues de convergence de séries multiples d'ordre m de nombres complexes et on démontre des théorèmes analogues. Entre autre le Théorème 1.2 est une version discrète du théorème sur les sommations successives des séries d'ordre m si celles-ci convergent régulièrement. Jusqu'à maintenant un tel théorème était connu uniquement pour des séries multiples d'ordre m absolument convergentes.

1. On the convergence of multiple series of complex numbers

We consider the m-multiple series

$$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_m=0}^{\infty} c_{j_1, j_2, \ldots, j_m} \tag{1}$$

of complex numbers, where $m \geqslant 2$ is a fixed integer. The m-tuple rectangular partial sums of (1) are defined by

$$s(l_1, l_2, \ldots, l_m) := \sum_{j_1=0}^{l_1} \sum_{j_2=0}^{l_2} \ldots \sum_{j_m=0}^{l_m} c_{j_1, j_2, \ldots, j_m}, \quad \text{where } (l_1, l_2, \ldots, l_m) \in \mathbb{N}^m.$$

We recall that (1) is said to converge in Pringsheim's sense to the sum $s \in \mathbb{C}$, in symbols: $\lim_{l_1, l_2, \ldots, l_m \to \infty} s(l_1, l_2, \ldots, l_m) = s$, if for every $\varepsilon > 0$ there exists $\lambda_1 = \lambda_1(\varepsilon) \in \mathbb{N}$ such that

$$\left|s(l_1, l_2, \ldots, l_m) - s\right| < \varepsilon \quad \text{if } \min\{l_1, l_2, \ldots, l_m\} > \lambda_1. \tag{2}$$

See [8] by Pringsheim, and also [10, p. 303, just after formula (1.18)] by Zygmund without indication of the term 'in Pringsheim's sense'.

In contrast to convergent single series, the convergence of multiple series in Pringsheim's sense implies neither the boundedness of the sequence $\{c_{j_1, j_2, \ldots, j_m}\colon (j_1, j_2, \ldots, j_m) \in \mathbb{N}^m\}$ of its terms, nor the convergence of its subseries. We recall that for any choice $1 \leqslant p_1 < p_2 < \cdots < p_e \leqslant m$ of integers, where $1 \leqslant e < m$, while denoting by $1 \leqslant p_{e+1} < \cdots < p_m \leqslant m$ the remaining integers between 1 and m, for any choice of the values $(j_{p_{e+1}}, \ldots, j_{p_m}) \in \mathbb{N}^{m-e}$, the e-multiple series

$$\sum_{j_{p_1}=0}^{\infty} \sum_{j_{p_2}=0}^{\infty} \ldots \sum_{j_{p_e}=0}^{\infty} c_{j_1, j_2, \ldots, j_m} \tag{3}$$

is called a subseries of (1). We refer to [6, Examples 1 and 2], where examples are given for the above mentioned deficiencies of the convergence notion in Pringsheim's sense in the case of double series.

Next, we recall the notion of convergence in the regular sense, or briefly: the regular convergence of the multiple series (1). Given any two pairs $(k_1, k_2, \ldots, k_m)$ and $(l_1, l_2, \ldots, l_m) \in \mathbb{N}^m$, where $0 \leqslant k_p \leqslant l_p$, $p = 1, 2, \ldots, m$; we set

$$s(k_1, l_1; k_2, l_2; \ldots; k_m, l_m) := \sum_{j_1=k_1}^{l_1} \sum_{j_2=k_2}^{l_2} \ldots \sum_{j_m=k_m}^{l_m} c_{j_1, j_2, \ldots, j_m}. \tag{4}$$

Now, we say that the multiple series (1) converges regularly if for every $\varepsilon > 0$ there exists $\lambda_2 = \lambda_2(\varepsilon) \in \mathbb{N}$ such that

$$\left|s(k_1,l_1;k_2,l_2;\ldots;k_m,l_m)\right| < \varepsilon \quad \text{if } \max\{k_1,k_2,\ldots,k_m\} > \lambda_2 \text{ and } 0 \leqslant k_p \leqslant l_p,\ p = 1,2,\ldots,m. \tag{5}$$

For double series, this notion was introduced in [3], where it was called as 'convergence in a restricted sense'; while for m-multiple series, $m \geqslant 3$, it was introduced in [4] by the present author.

We note that the term 'regular convergence' of a double series is due to Hardy [2]. According to his definition, the double series $\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} c_{j_1,j_2}$ is said to converge regularly if it converges in Pringsheim's sense and, in addition, each of its so-called row and column subseries defined by

$$\sum_{j_2=0}^{\infty} c_{j_1,j_2}, \quad \text{where } j_1 \in \mathbb{N}; \qquad \sum_{j_1=0}^{\infty} c_{j_1,j_2}, \quad \text{where } j_2 \in \mathbb{N};$$

converges as a single series. It is easy to see that the definition given by us in (5) for $m = 2$ is equivalent to the definition given by Hardy.

It is routine to check that if (5) is satisfied, then

$$\left|s(l_1,l_2,\ldots,l_m) - s(k_1,k_2,\ldots,k_m)\right| < m\varepsilon \quad \text{if } \min\{l_1,l_2,\ldots,l_m;k_1,k_2,\ldots,k_m\} > \rho_2(\varepsilon). \tag{6}$$

Since $\varepsilon > 0$ is arbitrary, by virtue of the Cauchy convergence criterion, the regular convergence of (1) implies its convergence in Pringsheim's sense. Consequently, the sum of a regularly convergent multiple series is well defined. On the other hand, a multiple series may converge in Pringsheim's sense, without converging regularly. See, e.g., [6, Example 3] in the case of a double series.

The following characterization of the regular convergence of a multiple series (1) was proved in [7]:

Theorem 1.1. *The m-multiple series* (1) *converges regularly if and only if* (i) *it converges in Pringsheim's sense*; *and* (ii) *each of its $(m-1)$-multiple subseries $\sum_{j_1=0}^{\infty}\cdots\sum_{j_{p-1}=0}^{\infty}\sum_{j_{p+1}=0}^{\infty}\cdots\sum_{j_m=0}^{\infty} c_{j_1,j_2,\ldots,j_m}$ converges regularly for all choices of $p \in \{1,2,\ldots,m\}$ and $j_p \in \mathbb{N}$.*

An immediate consequence of Theorem 1.1 is the following statement: The multiple series (1) converges regularly if and only if it converges in Pringsheim's sense, and each of its subseries defined in (3) converges regularly; or equivalently, each of its subseries converges in Pringsheim's sense; where by the regular convergence or by the convergence in Pringsheim's sense of a single series we mean its ordinary convergence.

It is obvious that if the multiple series (1) converges absolutely, then it converges also regularly. The converse statement is false in general. For example, see the double series in [6, Example 5], which converges regularly, but not absolutely.

The next theorem proved in [7] says that if the multiple series (1) converges regularly, then its sum can be also obtained by successive summation.

Theorem 1.2. *If the m-multiple series* (1) *converges regularly and $\{\sigma(1),\sigma(2),\ldots,\sigma(m)\}$ is any permutation of $\{1,2,\ldots,m\}$, then*

$$\sum_{j_{\sigma(1)}=0}^{\infty}\left(\sum_{j_{\sigma(2)}=0}^{\infty}\left(\cdots\left(\sum_{j_{\sigma(m)}=0}^{\infty} c_{j_1,j_2,\ldots,j_m}\right)\cdots\right)\right) = s,$$

where s is the sum of (1) *in Pringsheim's sense.*

In Harmonic Analysis (see, e.g., multiple Fourier series), m-multiple series of the form

$$\sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty}\cdots\sum_{j_m=-\infty}^{\infty} c_{j_1,j_2,\ldots,j_m} \tag{7}$$

frequently occur. Using symmetric rectangular partial sums defined by

$$s(l_1,l_2,\ldots,l_m) := \sum_{j_1=-l_1}^{l_1}\sum_{j_2=-l_2}^{l_2}\cdots\sum_{j_m=-l_m}^{l_m} c_{j_1,j_2,\ldots,j_m}, \quad \text{where } (l_1,l_2,\ldots,l_m) \in \mathbb{N}^m, \tag{8}$$

the convergence of (7) in Pringsheim's sense is also defined by (2).

Instead of definition (4), using the following one:

$$s(k_1,l_1;k_2,l_2;\ldots;k_m,l_m) := \sum_{k_1 \leqslant |j_1| \leqslant l_1}\sum_{k_2 \leqslant |j_2| \leqslant l_2}\cdots\sum_{k_m \leqslant |j_m| \leqslant l_m} c_{j_1,j_2},\ldots,j_m, \tag{9}$$

the regular convergence of (7) is also defined by (5).

2. Convergence of multiple integrals of locally integrable functions

Let $f : \overline{\mathbb{R}}^m_+ \to \mathbb{C}$ be a locally integrable function in Lebesgue's sense over the nonnegative octant $\overline{\mathbb{R}}^m_+ := [0,\infty)^m$, in symbols: $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^m_+)$, where $m \geqslant 2$ is a fixed integer. We consider the m-multiple integral

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty f(t_1,t_2,\ldots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\ldots\mathrm{d}t_m, \tag{10}$$

whose m-tuple rectangular partial integrals are defined by

$$I(v_1,v_2,\ldots,v_m) := \int_0^{v_1}\int_0^{v_2}\ldots\int_0^{v_m} f(t_1,t_2,\ldots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\ldots\mathrm{d}t_m, \quad \text{where } (v_1,v_2,\ldots,v_m) \in \overline{\mathbb{R}}^m_+. \tag{11}$$

Analogously to the convergence of multiple series in Pringsheim's sense (cf. (2)), the multiple integral (10) is said to converge in Pringsheim's sense to the limit $I \in \mathbb{C}$, or equivalently, we may say that this I is the value of (10) in Pringsheim's sense, in symbols: $\lim_{v_1,v_2,\ldots,v_m\to\infty} I(v_1,v_2,\ldots,v_m) = I$, if for every $\varepsilon > 0$ there exists $\rho_1 = \rho_1(\varepsilon) \in \mathbb{R}_+$ such that

$$\bigl|I(v_1,v_2,\ldots,v_m) - I\bigr| < \varepsilon \quad \text{if } \min\{v_1,v_2,\ldots,v_m\} > \rho_1. \tag{12}$$

Next, we recall the notion of regular convergence for m-multiple integrals (see [5] in the case $m = 2$, and [7] in the case $m \geqslant 3$, by the present author). To this effect, given any two pairs $(u_1,u_2,\ldots,u_m)$ and $(v_1,v_2,\ldots,v_m) \in \overline{\mathbb{R}}^m_+$, where $0 \leqslant u_k \leqslant v_k$, $k = 1,2,\ldots,m$; we set (cf. (4))

$$I(u_1,v_1;u_2,v_2;\ldots;u_m,v_m) := \int_{u_1}^{v_1}\int_{u_2}^{v_2}\ldots\int_{u_m}^{v_m} f(t_1,t_2,\ldots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\ldots\mathrm{d}t_m. \tag{13}$$

Now, we say that the multiple integral (10) converges regularly if for every $\varepsilon > 0$ there exists $\rho_2 = \rho_2(\varepsilon) \in \mathbb{R}_+$ such that (cf. (5)).

$$\bigl|I(u_1,v_1;u_2,v_2;\ldots;u_m,v_m)\bigr| < \varepsilon \quad \text{if } \max\{u_1,u_2,\ldots,u_m\} > \rho_2 \text{ and } 0 \leqslant u_k \leqslant v_k,\ k = 1,2,\ldots,m. \tag{14}$$

It is routine to check that if (14) is satisfied, then (cf. (6))

$$\bigl|I(v_1,v_2,\ldots,v_m) - I(u_1,u_2,\ldots,u_m)\bigr| < m\varepsilon \quad \text{if } \min\{v_1,v_2,\ldots,v_m;u_1,u_2,\ldots,u_m\} > \rho_2(\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, by virtue of the Cauchy convergence criterion, the regular convergence of (10) implies its convergence in Pringsheim's sense. Consequently, the value of a regularly convergent multiple integral is well defined. On the other hand, a multiple integral may converge in Pringsheim's sense, without converging regularly. See, e.g., [6, just before Example 6] in the case of a double integral.

It is obvious that if $f \in L^1(\overline{\mathbb{R}}^m_+)$, then the multiple integral (10) converges regularly and its value is equal to the Lebesgue integral of f over the whole octant $\overline{\mathbb{R}}^m_+$. The converse statement is not true in general. See, e.g., [6, Example 6] in the case of a double integral, which converges regularly, but $f \notin L^1(\overline{\mathbb{R}}^m_+)$.

For the sake of brevity in writing, we present a characterization of the regular convergence of the m-multiple integral (10) in the case $m = 2$. It is clear that if $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$ and the double integral

$$\int_0^\infty\int_0^\infty f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{15}$$

converges regularly, then the finite limit of the so-called 'horizontal strip' integral

$$\lim_{x_1\to\infty}\int_0^{x_1}\int_y^{y_1} f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{16}$$

exists locally uniformly in (y,y_1), where $0 \leqslant y \leqslant y_1$. By the term 'locally uniform convergence', in this case we mean that for every $c > 0$ and $\varepsilon > 0$, there exists $\rho_3 = \rho_3(c,\varepsilon) \in \mathbb{R}_+$ such that

$$\left|\int_0^{x_1}\int_y^{y_1} f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2\right| < \varepsilon \quad \text{for all } 0 \leqslant y \leqslant y_1 \leqslant c \quad \text{if } x_1 > \rho_3.$$

Analogously, if the double integral (15) converges regularly, then the finite limit of the so-called 'vertical strip' integral

$$\lim_{y_1\to\infty}\int_x^{x_1}\int_0^{y_1} f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{17}$$

exists locally uniformly in (x, x_1), where $0 \leqslant x \leqslant x_1$.

Now, the following characterization of regular convergence in the case of double integrals was given in [6]:

Theorem 2.1. *Suppose $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$. Then the double integral* (15) *converges regularly if and only if*

(i) *it converges in Pringsheim's sense*; *and*
(ii) *the finite limit of the 'horizontal strip' integral in* (16) *exists locally uniformly in (y, y_1), where $0 \leqslant y \leqslant y_1$; as well as the finite limit of the 'vertical strip' integral in* (17) *also exists locally uniformly in (x, x_1), where $0 \leqslant x \leqslant x_1$.*

We note that the sufficiency part in Theorem 2.1 fails if we drop the requirement 'locally uniformly' in condition (ii). In [6, Example 7], we presented a function $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^m_+)$ such that the double series (15) converges to 0 in Pringsheim's sense, all its 'horizontal strip' integrals as well as all its 'vertical strip' integrals converge to 0, but not locally uniformly, and (15) fails to converge regularly.

In Harmonic Analysis (see, e.g., multiple Fourier transforms), multiple integrals of the form (cf. (7))

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} f(t_1,t_2,\dots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\dots\mathrm{d}t_m \tag{18}$$

frequently occur, where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Using symmetric rectangular partial integrals defined by

$$I(v_1,v_2,\dots,v_m) := \int_{-v_1}^{v_1}\int_{-v_2}^{v_2}\dots\int_{-v_m}^{v_m} f(t_1,t_2,\dots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\dots\mathrm{d}t_m,\quad \text{where } (v_1,v_2,\dots,v_m)\in\overline{\mathbb{R}}^m_+$$

(cf. (8)), the convergence of (18) in Pringsheim's sense is defined also by (12).

Instead of definition (13), using the following one:

$$I(u_1,v_1;u_2,v_2;\dots;u_m,v_m) := \int_{u_1<|t_1|<v_1}\int_{u_2<|t_2|<v_2}\dots\int_{u_m<|t_m|<v_m} f(t_1,t_2,\dots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\dots\mathrm{d}t_m,$$
$$\text{where } 0\leqslant u_k\leqslant v_k,\ k=1,2,\dots,m;$$

(cf. (9)), the regular convergence of (18) is also defined by (14).

3. A generalized version of Fubini's theorem

We recall Fubini's classical theorem on successive integration of the m-multiple integral (10) (see [1] by Fubini, and see also, e.g., [9, p. 85] by F. Riesz and B. Sz.-Nagy): If $f \in L^1(\overline{\mathbb{R}}^m_+)$ and $m = p + q$, where $p, q \in \mathbb{N}_+$, then

$$\int_{\overline{\mathbb{R}}^m_+} f(t_1,t_2,\dots,t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2\dots\mathrm{d}t_m = \int_{\overline{R}^p_+}\left(\int_{\overline{\mathbb{R}}^q_+} f(t_1,t_2,\dots,t_m)\,\mathrm{d}t_{p+1}\,\mathrm{d}t_{p+2}\dots\mathrm{d}t_m\right)\mathrm{d}t_1\,\mathrm{d}t_2\dots\mathrm{d}t_p, \tag{19}$$

where the inner integral exists in Lebesgue's sense for almost every $(t_1,t_2,\dots,t_p) \in \overline{\mathbb{R}}^p_+$, and the outer integral also exists in Lebesgue's sense.

It follows from (19) that Fubini's theorem holds true even in the following more general form: If $f \in L^1(\overline{\mathbb{R}}^m_+)$ and $\{\sigma(1),\sigma(2),\dots,\sigma(m)\}$ is a permutation of $\{1,2,\dots,m\}$, then the left-hand side in (19) can be also obtained by the repeated successive integration

$$\int_0^\infty\left(\int_0^\infty\left(\dots\left(\int_0^\infty f(t_1,t_2,\dots,t_m)\,\mathrm{d}t_{\sigma(m)}\right)\dots\right)\mathrm{d}t_{\sigma(2)}\right)\mathrm{d}t_{\sigma(1)}. \tag{20}$$

Our main result proved in [7] reads as follows.

Theorem 3.1. *If $f \in L^1_{\text{loc}}(\overline{\mathbb{R}}^m_+)$, $m = p + q$ where $p, q \in \mathbb{N}_+$, and the m-multiple integral converges regularly, then*

(i) *the finite limit*

$$\lim_{v_{p+1}, v_{p+2}, \ldots, v_m \to \infty} I(u_1, v_1; u_2, v_2; \ldots; u_p, v_p; 0, v_{p+1}; 0, v_{p+2}; \ldots; 0, v_m) =: J(u_1, v_1; u_2, v_2; \ldots; u_p, v_p),$$
$$\text{where } 0 \leqslant u_k \leqslant v_k,\ k = 1, 2, \ldots, p;\ \text{exists uniformly in each of its variables}; \tag{21}$$

(ii) *the limit function $J : \overline{\mathbb{R}}^{2p}_+ \to \mathbb{C}$ is such that for every $\varepsilon > 0$ there exists $\rho_3 = \rho_3(\varepsilon) \in \mathbb{R}_+$ such that*

$$\left|J(u_1, v_1; u_2, v_2; \ldots; u_p, v_p)\right| < \varepsilon \quad \text{if } \max\{u_1, u_2, \ldots, u_p\} > \rho_3 \text{ and } 0 \leqslant u_k \leqslant v_k,\ k = 1, 2, \ldots, p; \tag{22}$$

(iii) *the finite limit*

$$\lim_{v_1, v_2, \ldots, v_p \to \infty} J(0, v_1; 0, v_2; \ldots; 0, v_p) =: J(0, \infty; 0; \infty; \ldots; 0, \infty) \tag{23}$$

also exists and it is equal to I, where I is the value of the multiple integral (10) *in Pringsheim's sense.*

In the case of double integrals, we refer to our Abstract, where Theorem 3.1 is presented in a more transparent form.

A trivial consequence of Theorem 3.1 is that if $f \in L^1_{\text{loc}}(\overline{\mathbb{R}}^m_+)$, $m = p + q$ where $p, q \in \mathbb{N}_+$, the m-multiple integral (10) converges regularly, and $f(t_1, t_2, \ldots, t_p, t_{p+1}, \ldots, t_m) \in L^1(\overline{\mathbb{R}}^q_+)$ for almost every $(t_1, t_2, \ldots, t_p) \in \overline{\mathbb{R}}^p_+$, then we have

$$J(u_1, v_1; u_2, v_2; \ldots; u_p, v_p) = \int_{u_1}^{v_1}\int_{u_2}^{v_2} \ldots \int_{u_p}^{v_p}\left(\int_{\overline{\mathbb{R}}^q_+} f(t_1, t_2, \ldots, t_p, t_{p+1}, t_{p+2}, \ldots, t_m) \times \mathrm{d}t_{p+1}\,\mathrm{d}t_{p+2}, \ldots \mathrm{d}t_m\right)\mathrm{d}t_1\,\mathrm{d}t_2 \ldots \mathrm{d}t_p, \quad \text{where } 0 \leqslant u_k \leqslant v_k,\ k = 1, 2, \ldots, p.$$

We observe that the limit $J = J(u_1, v_1; u_2, v_2; \ldots; u_p, v_p)$ in (21) is an additive function of the p-tuple rectangle $\mathcal{R} := [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_p, v_p]$, due to the additivity property of the Lebesgue integral; and $J = 0$ if $0 \leqslant u_k = v_k$ for some $1 \leqslant k \leqslant p$. Keeping in mind the definition of regular convergence introduced for multiple integrals by the requirement (14), inequality (22) can be interpreted as the regular convergence of $J(0, \infty; 0, \infty; \ldots; 0, \infty)$ defined in (23).

A more important consequence of Theorem 3.1 is the following one. If we apply Theorem 3.1 in the case when $p := m - 1$ and $q := 1$; then we repeat the 'successive integration' for $J(u_1, v_1; u_2, v_2; \ldots; u_{m-1}, v_{m-1}) : \overline{\mathbb{R}}^{2(p-1)}_+ \to \mathbb{C}$ in the case when $p := m - 2$ and $q := 1$; and so on, repeating this process altogether $(m - 1)$ times, we get to a counterpart of (20) in the case when $\sigma(k) = k$ for $k = 1, 2, \ldots, m$.

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