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Partial Differential Equations

A kinetic eikonal equation

Une équation eikonale cinétique

Emeric Bouin, Vincent Calvez

UMR CNRS 5669 'UMPA' and INRIA project 'NUMED', École normale supérieure de Lyon, 46, allée d'Italie, 69364 Lyon cedex 07, France

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ABSTRACT

We analyse the linear kinetic transport equation with a BGK relaxation operator. We study the large scale hyperbolic limit $(t,x) \to (t/\varepsilon, x/\varepsilon)$. We derive a new type of limiting Hamilton–Jacobi equation, which is analogous to the classical eikonal equation derived from the heat equation with small diffusivity. Interestingly, the hydrodynamic limit and the large deviation approach do not commute. We prove well-posedness of the phase problem and convergence towards the viscosity solution of the Hamilton–Jacobi equation. This is a preliminary work before analyzing the propagation of reaction fronts in kinetic equations.

RÉSUMÉ

Nous analysons une équation cinétique linéaire de transport avec un opérateur de relaxation BGK. Nous étudions la limite hyperbolique de grande échelle $(t,x) \to (t/\varepsilon, x/\varepsilon)$. Nous obtenons à la limite une nouvelle équation de Hamilton–Jacobi, qui est l'analogue de l'équation eikonale classique obtenue à partir de l'équation de la chaleur avec petite diffusion. Il est alors intéressant de constater que la limite hydrodynamique ne commute pas avec l'asymptotique des grandes déviations. Nous démontrons le caractère bien posé de l'équation vérifiée par la phase, ainsi que la convergence vers une solution de viscosité de l'équation de Hamilton–Jacobi. Ceci est un travail préliminaire en vue d'analyser la propagation de fronts de réaction pour des équations cinétiques.

Version française abrégée

Nous considérons un modèle cinétique linéaire avec un opérateur de relaxation BGK, posé sur un ensemble de vitesses V symétrique et borné. On analyse le comportement de l'équation dans la limite hyperbolique de grande échelle $(t,x) \to (\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$,

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\big(M(v)\rho^\varepsilon - f^\varepsilon\big), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V. \tag{1}$$

Nous démontrons que la phase φ^ε définie par la relation $f^\varepsilon(t,x,v) = M(v)e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}$ converge (localement) uniformément, lorsque $\varepsilon \to 0$, vers une fonction $\varphi^0(t,x)$ indépendante de v. De surcroît, la fonction φ^0 est solution de viscosité

E-mail addresses: emeric.bouin@ens-lyon.fr (E. Bouin), vincent.calvez@ens-lyon.fr (V. Calvez).

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de l'équation de Hamilton–Jacobi suivante,

$$\int_V \frac{M(v)}{1-\partial_t\varphi^0(t,x)-v\cdot\nabla_x\varphi^0(t,x)}\,\mathrm{d}v=1,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n. \tag{2}$$

Cette équation peut se réécrire sous la forme canonique $\partial_t\varphi^0+H(\nabla_x\varphi^0)=0$ pour un hamiltonien effectif $H(p)$ qui est lipschitzien et convexe. Comme dans [1], cet hamiltonien est relié à la résolution d'un problème aux valeurs propres dans l'espace des vitesses V, ce dernier s'écrivant comme suit : trouver un vecteur propre $Q(v)$ et une valeur propre $H(p)$ tels que

$$\big(1+H(p)-v\cdot p\big)Q(v)=\int_V M\big(v'\big)Q\big(v'\big)\,\mathrm{d}v'.$$

La démonstration du passage à la limite de (1) vers (2) s'appuie sur une série d'estimations *a priori* qui démontre que φ^ε appartient à l'espace de Sobolev $W^{1,\infty}(\mathbb{R}_+\times\mathbb{R}^n\times V)$, avec un contrôle uniforme en $\varepsilon>0$ (Proposition 2.1 ci-dessous). Dans un deuxième temps, nous démontrons que toute fonction test $\psi^0(t,x)$ de classe $\mathcal{C}^2$ telle que $\varphi^0-\psi^0$ admet un maximum local en (t^0,x^0), vérifie

$$\int_V \frac{M(v)}{1-\partial_t\psi^0(t^0,x^0)-v\cdot\nabla_x\psi^0(t^0,x^0)}\,\mathrm{d}v\leqslant 1.$$

Ceci démontre que φ^0 est une sous-solution de viscosité l'équation de Hamilton–Jacobi (2). Un raisonnement identique montre qu'il s'agit aussi d'une sur-solution de viscosité. La démonstration se base sur la construction d'un correcteur microscopique $\eta(t,x,v)$ défini de façon ad-hoc par la relation

$$\forall\big(v,v'\big)\in V\times V,\quad e^{\eta(t,x,v)}-e^{\eta(t,x,v')}=\big(v'-v\big)\cdot\nabla_x\psi^0(t,x).$$

1. Large scale limit and derivation of the Hamilton–Jacobi equation

We consider the following kinetic equation with BGK relaxation operator:

$$\partial_t f+v\cdot\nabla_x f=M(v)\rho-f,\quad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^n\times V, \tag{3}$$

where $f(t,x,v)$ denotes the density of particles moving with speed $v\in V$ at time t and position x. The function $\rho(t,x)$ denotes the macroscopic density of particules:

$$\rho(t,x)=\int_V f(t,x,v)\,\mathrm{d}v,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n.$$

Here V denotes a bounded symmetric subset of $\mathbb{R}^n$. We assume that the Maxwellian M is symmetric and satisfies the following moment identities:

$$\int_V M(v)\,\mathrm{d}v=1,\qquad \int_V vM(v)\,\mathrm{d}v=0,\qquad \int_V v^2M(v)\,\mathrm{d}v=\theta^2.$$

In this paper we focus on the large scale hyperbolic limit $(t,x)\to(\frac{t}{\varepsilon},\frac{x}{\varepsilon})$, $\varepsilon\to 0$. The kinetic equation (3) reads as follows in the new scaling:

$$\partial_t f^\varepsilon+v\cdot\nabla_x f^\varepsilon=\frac{1}{\varepsilon}\big(M(v)\rho^\varepsilon-f^\varepsilon\big),\quad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^n\times V. \tag{4}$$

Clearly, the velocity distribution relaxes rapidly towards the Maxwellian distribution. This motivates the introduction of the following Hopf–Cole transformation:

$$f^\varepsilon(t,x,v)=M(v)e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}},$$

where we expect the phase φ^ε to become independent of v as $\varepsilon\to 0$. To avoid technical complications due to ill-prepared data, we set $\varphi^\varepsilon(0,x,v)=\varphi_0(x)\geqslant 0$ as an initial data for (4). The equation satisfied by φ^ε reads

$$\partial_t\varphi^\varepsilon+v\cdot\nabla_x\varphi^\varepsilon=\int_V M\big(v'\big)\big(1-e^{\frac{\varphi^\varepsilon-\varphi^{\varepsilon\prime}}{\varepsilon}}\big)\,\mathrm{d}v',\quad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^n\times V. \tag{5}$$

Theorem 1.1. *Let $V \subset \mathbb{R}^n$ be bounded and symmetric, and $M \in L^1(V)$ be nonnegative and symmetric. Then φ^ε converges (locally) uniformly towards φ^0, where φ^0 does not depend on v. Moreover φ^0 is the viscosity solution of the following Hamilton–Jacobi equation*:

$$\int_V \frac{M(v)}{1 - \partial_t \varphi^0(t,x) - v \cdot \nabla_x \varphi^0(t,x)}\,\mathrm{d}v = 1, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{6}$$

The denominator of the integrand is positive for all $v \in V$.

The last observation in Theorem 1.1 is not compatible with an unbounded velocity set.

As in [1,11], the Hamilton–Jacobi equation is connected with an eigenvalue problem in the velocity space V: Find an eigenvector $Q(v)$ such that

$$\left(1 - \partial_t \varphi^0 - v \cdot \nabla_x \varphi^0\right) Q(v) = \int_V M(v') Q(v')\,\mathrm{d}v'.$$

This eigenproblem can be solved explicitly, and yields formula (6).

Thanks to monotonicity properties, we can boil down to the classical framework of first order Hamilton–Jacobi equations. Indeed, writing Eq. (6) as $G(\partial_t \varphi^0, \nabla_x \varphi^0) = 0$, we observe that G is increasing with respect to the first variable. Hence the equation is equivalent to $\partial_t \varphi^0 + H(\nabla_x \varphi^0) = 0$, where the effective Hamiltonian H is defined through the implicit formula,

$$\int_V \frac{M(v)}{(1 + H(p) - v \cdot p)}\,\mathrm{d}v = 1. \tag{7}$$

Differentiating (7) we obtain,

$$\int_V \frac{M(v)}{(1 + H(p) - v \cdot p)^2} \left(\nabla H(p) - v\right)\mathrm{d}v = 0.$$

We deduce $\|\nabla H\|_\infty \leqslant V_{\max}$. This is in accordance with the underlying kinetic equation, since ∇H can be interpreted as the group speed, which is bounded by the maximal speed of the particles. Differentiating (7) twice we obtain

$$\left(\int_V \frac{M(v)}{(1 + H(p) - v \cdot p)^2}\,\mathrm{d}v\right) \mathrm{D}^2 H(p) = 2 \int_V \frac{M(v)}{(1 + H(p) - v \cdot p)^3} \left(\nabla H(p) - v\right) \otimes \left(\nabla H(p) - v\right) \mathrm{d}v.$$

We deduce that the effective Hamiltonian is convex.

As an example, we can compute the effective Hamiltonian H in one dimension for a constant Maxwellian $M \equiv \frac{1}{2}$ on $V = (-1,1)$. We obtain $H(p) = \frac{p - \tanh(p)}{\tanh(p)}$. It is equivalent to $\theta^2 |p|^2$ for small p ($\theta^2 = \frac{1}{3}$). Another example where the effective Hamitonian is explicit is given by the Maxwellian $M(v) = \frac{1}{2}(\delta_1 + \delta_{-1})$, though it is not an L^1 function. This corresponds to a two velocities model (also known as the *telegraph equation*, see [10,3]). In this case we obtain the relativistic Hamiltonian $H(p) = \frac{\sqrt{1+4p^2} - 1}{2}$.

Interestingly enough, we obtain a Hamilton–Jacobi equation which differs from the classical eikonal equation. The latter could have been expected from the following argumentation. The formal limit of Eq. (4) at order $O(\varepsilon)$ is the heat equation with small diffusivity:

$$\partial_t \rho^\varepsilon = \varepsilon \theta^2 \Delta_x \rho^\varepsilon, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

It is well known that the phase $\phi^\varepsilon = -\varepsilon \log \rho^\varepsilon$ satisfies in the limit $\varepsilon \to 0$ the classical eikonal equation in the sense of viscosity solutions [8,12,6,9,13,11]:

$$\partial_t \phi^0 + \theta^2 \left|\nabla_x \phi^0\right|^2 = 0. \tag{8}$$

Interestingly, the hydrodynamic limit and the large deviation approach do not commute. We only have asymptotic equivalence between the two approaches for small $|p|$ as can be seen directly on (7) by Taylor expansion: $H(p) \sim \theta^2 |p|^2$.

In Fig. 1 we show numerical simulations of the kinetic eikonal equation (6), with a constant Maxwellian on $V = (-1,1)$, and we compare it with the classical eikonal equation (8).

We end this introduction by listing some possible extensions of Theorem 1.1 for other choices of transport and scattering operators. We will develop a more general framework in a future work.

(i) In the case of the Vlasov–Fokker–Planck equation,

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon - \nabla_x V(x) \cdot \nabla_v f^\varepsilon = \frac{1}{\varepsilon} \nabla_v \cdot \left(\nabla_v f^\varepsilon + v f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n,$$

we obtain simply the eikonal equation $\partial_t \phi^0 + |\nabla_x \phi^0|^2 = 0$ in the WKB expansion $f^\varepsilon = M(v) e^{-\frac{\varphi^\varepsilon}{\varepsilon}}$, where $M(v)$ is a Gaussian.

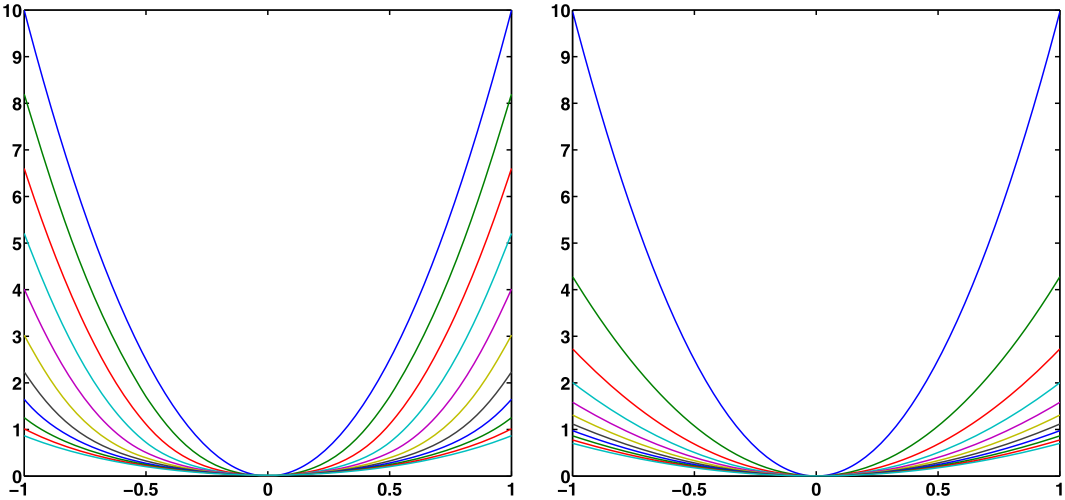

Fig. 1. Numerical simulations of the Hamilton–Jacobi equation $\partial_t\varphi + H(\partial_x\varphi) = 0$. (*Left*) The kinetic eikonal equation (6) where $M(v) = \frac{1}{2}\mathbf{1}_{(-1,1)}$. (*Right*) The classical eikonal equation $H(p) = \theta^2|p|^2$ (8). In both cases the initial condition is a parabola.

(ii) It is challenging to replace the BGK operator in (4) by a convolution operator $L(f) = K * f - f$, where K is a probability kernel [2]. However in this case we are not able to solve explicitly the eigenproblem in the cell V.

(iii) In a forthcoming work we will investigate the propagation of reaction fronts in kinetic equations, following [9,10,5,3].

2. Proof of Theorem 1.1

First let us mention that the solution φ^ε remains nonnegative for all times. We proceed in two steps. First we prove uniform estimates with respect to $\varepsilon > 0$. It allows to extract a uniformly converging subsequence. Second we identify the limit as the viscosity solution of Eq. (6) using the maximum principle. The second step relies on the construction of a suitable corrector $\eta(t, x, v)$ [6,7].

Step 1. Existence and uniform bounds.

Proposition 2.1. *Let $V \subset \mathbb{R}^n$ be a bounded subset. Assume $M \in L^1(V)$ and $\varphi_0 \in W^{1,\infty}(\mathbb{R}^n)$. The kinetic equation* (5) *has a unique solution $\varphi^\varepsilon \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times V)$. Furthermore, the solution satisfies the following uniform estimates*:

$$0 \leqslant \varphi^\varepsilon(t,\cdot) \leqslant \|\varphi_0\|_\infty, \tag{9}$$

$$\left\|\nabla_x\varphi^\varepsilon(t,\cdot)\right\|_\infty \leqslant \|\nabla_x\varphi_0\|_\infty, \tag{10}$$

$$\left\|\nabla_v\varphi^\varepsilon(t,\cdot)\right\|_\infty \leqslant t\|\nabla_x\varphi_0\|_\infty, \tag{11}$$

$$\left\|\partial_t\varphi^\varepsilon(t,\cdot)\right\|_\infty \leqslant V_{\max}\|\nabla_x\varphi_0\|_\infty. \tag{12}$$

Proof. We obtain a unique solution φ^ε from a fixed point method on the Duhamel formulation of (5):

$$\varphi^\varepsilon(t,x,v) = \varphi_0(x - tv) + \int_0^t \int_V M(v')\left(1 - e^{\frac{\varphi^\varepsilon(t-s,x-sv,v)-\varphi^\varepsilon(t-s,x-sv,v')}{\varepsilon}}\right) \mathrm{d}v'\,\mathrm{d}s. \tag{13}$$

We obtain directly,

$$\forall\varepsilon > 0, \quad 0 \leqslant \varphi^\varepsilon(t,x,v) \leqslant \varphi_0(x - tv) + t.$$

This ensures that φ^ε is uniformly bounded on $[0,T] \times \mathbb{R}^n \times V$. To prove the bound (9), we define $\psi_\delta^\varepsilon(t,x,v) = \varphi^\varepsilon(t,x,v) - \delta t - \delta^4|x|^2$. For any $\delta > 0$, ψ_δ^ε attains a maximum at point $(t_\delta, x_\delta, v_\delta)$. Suppose that $t_\delta > 0$. Then, we have

$$\partial_t\varphi^\varepsilon(t_\delta,x_\delta,v_\delta) \geqslant \delta, \quad \nabla_x\varphi^\varepsilon(t_\delta,x_\delta,v_\delta) = 2\delta^4 x_\delta.$$

As a consequence, we have at the maximum point $(t_\delta, x_\delta, v_\delta)$:

$$0 \geqslant \int_V M(v')\left(1 - e^{\frac{\psi_\delta^\varepsilon(t_\delta,x_\delta,v_\delta)-\psi_\delta^\varepsilon(t_\delta,x_\delta,v')}{\varepsilon}}\right)\mathrm{d}v' \geqslant \delta + 2v_\delta\delta^4 x_\delta \geqslant \delta - 2V_{\max}\delta^4|x_\delta|. \tag{14}$$

Moreover, the maximal property of $(t_\delta, x_\delta, v_\delta)$ also implies

$$\|\varphi^\varepsilon\|_\infty - \delta^4|x_\delta|^2 \geqslant \varphi^\varepsilon(t_\delta, x_\delta, v_\delta) - \delta t_\delta - \delta^4|x_\delta|^2 \geqslant \varphi^\varepsilon(0,0,v_\delta) \geqslant 0.$$

We obtain a contradiction with (14) since $\delta^{-6}/(2V_{\max}) \leqslant |x_\delta|^2 \leqslant \delta^{-4}\|\varphi^\varepsilon\|_\infty$ cannot hold for sufficiently small $\delta > 0$. As a consequence $t_\delta = 0$, and we have,

$$\forall(t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \quad \varphi^\varepsilon(t,x,v) \leqslant \varphi^0(x_\delta, v_\delta) + \delta t + \delta^4|x|^2 \leqslant \|\varphi_0\|_\infty + \delta t + \delta^4|x|^2.$$

Passing to the limit $\delta \to 0$, we obtain (9). To find the bound (10), we use the same ideas on the difference $\varphi_h^\varepsilon(t,x,v) = \varphi^\varepsilon(t,x+h,v) - \varphi^\varepsilon(t,x,v)$. The equation for φ_h^ε reads as follows

$$\partial_t\varphi_h^\varepsilon + v \cdot \nabla_x\varphi_h^\varepsilon = \int_V M(v')e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}}\left(1 - e^{\frac{\varphi_h^\varepsilon - \varphi_h^{\varepsilon\prime}}{\varepsilon}}\right)\mathrm{d}v'.$$

Using the same argument as above with a δ-correction, we conclude that

$$\forall(t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \quad \varphi_h^\varepsilon(t,x,v) \leqslant \sup_{(x,v)\in\mathbb{R}\times V}\left|\varphi^0(x+h,v) - \varphi^0(x,v)\right|.$$

The same argument applies to $-\varphi_h^\varepsilon$,

$$\partial_t\left(-\varphi_h^\varepsilon\right) + v \cdot \nabla_x\left(-\varphi_h^\varepsilon\right) = -\int_V M(v')e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}}\left(1 - e^{-\frac{(-\varphi_h^\varepsilon) - (-\varphi_h^{\varepsilon\prime})}{\varepsilon}}\right)\mathrm{d}v',$$

so that the r.h.s. has the right sign when $-\varphi_h^\varepsilon$ attains a maximum. Finally,

$$\forall(t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \quad \left|\varphi_h^\varepsilon(t,x,v)\right| \leqslant \sup_{(x,v)\in\mathbb{R}\times V}\left|\varphi^0(x+h,v) - \varphi^0(x,v)\right| \leqslant \left\|\nabla_x\varphi^0\right\|_\infty |h|,$$

from which the estimate (10) follows.

To obtain regularity in the velocity variable (11), we differentiate (5) with respect to v,

$$(\partial_t + v \cdot \nabla_x)\left(\nabla_v\varphi^\varepsilon\right) = -g_\varepsilon\left(\varphi^\varepsilon\right)\nabla_v\varphi^\varepsilon - \nabla_x\varphi^\varepsilon,$$

where $g_\varepsilon(\varphi^\varepsilon) = \frac{1}{\varepsilon}\int_V M(v')e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}}\,\mathrm{d}v' \geqslant 0$. Multiplying by $\frac{\nabla_v\varphi^\varepsilon}{|\nabla_v\varphi^\varepsilon|}$, we obtain

$$\begin{aligned}(\partial_t + v \cdot \nabla_x)\left(\left|\nabla_v\varphi^\varepsilon\right|\right) &= -g_\varepsilon\left(\varphi^\varepsilon\right)\left|\nabla_v\varphi^\varepsilon\right| - \left(\nabla_x\varphi^\varepsilon \cdot \frac{\nabla_v\varphi^\varepsilon}{|\nabla_v\varphi^\varepsilon|}\right)\\ &\leqslant \|\nabla_x\varphi_0\|_\infty,\end{aligned} \tag{15}$$

from which we deduce (11) since $\nabla_v\varphi_0 = 0$ by hypothesis.

Finally, the bound (12) is obtained similarly as the bound on $\nabla_x\varphi^\varepsilon$ (10), using the difference $\varphi_s^\varepsilon(t,x,v) = \varphi^\varepsilon(t+s,x,v) - \varphi^\varepsilon(t,x,v)$. We obtain

$$\forall(t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \quad \left|\varphi_s^\varepsilon(t,x,v)\right| \leqslant \sup_{(x,v)\in\mathbb{R}\times V}\left|\varphi^\varepsilon(s,x,v) - \varphi^0(x,v)\right|.$$

We use the Duhamel formulation (13) to estimate the last contribution:

$$\left|\varphi^\varepsilon(s,x,v) - \varphi^0(x,v)\right| \leqslant \left|\varphi_0(x - sv) - \varphi_0(x)\right| + o(s).$$

The estimate (12) follows. □

Step 2. Viscosity solution procedure.

From Proposition 2.1 we deduce that the family $(\varphi^\varepsilon)_\varepsilon$ is locally uniformly bounded in $W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times V)$. Then, from the Ascoli–Arzelà theorem, we can extract a locally uniformly converging subsequence. We denote by φ^0 the limit. Furthermore, from the fact that $\int_V M(v')e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}}\,\mathrm{d}v'$ is uniformly bounded on $[0,T] \times \mathbb{R}^n \times V$, we deduce that φ^0 does not depend on v.

Let $\psi^0 \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}^n)$ be a test function such that $\varphi^0 - \psi^0$ has a local maximum at (t^0, x^0). We want to show that ψ^0 is a subsolution of (6), yielding that φ^0 is a viscosity subsolution [4]. The supersolution case can be performed similarly. Thereby, we define a corrective term η not depending on ε [6]: $\psi^\varepsilon = \psi^0 + \varepsilon\eta$. The corrector η is defined up to an additive constant. We choose the renormalization $\int_V M(v')e^{-\eta'}\,\mathrm{d}v' = 1$. We define η as follows

$$\forall\left(v,v'\right) \in V \times V, \quad e^{\eta(t,x,v)} - e^{\eta(t,x,v')} = \left(v' - v\right) \cdot \nabla_x\psi^0(t,x). \tag{16}$$

The corrector η is well defined. In fact, we can choose any $v_0 \in V$ and define $e^{\eta(t,x,v)} = \mu_0 + (v_0 - v) \cdot \nabla_x \psi^0(t,x)$. There is a unique positive $\mu_0 = e^{\eta(t,x,v_0)}$ under the condition $\int_V M(v')e^{-\eta'}\,\mathrm{d}v' = 1$.

The uniform convergence ensures that $\varphi^\varepsilon - \psi^\varepsilon$ has a maximum at $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$, where $(t^\varepsilon, x^\varepsilon)$ is close to (t^0, x^0). As V is a bounded set, the sequence (v^ε) has an accumulation point, say v^*. We can extract a subsequence (without relabelling) such that $(t^\varepsilon, x^\varepsilon, v^\varepsilon) \to (t^0, x^0, v^*)$. We have at $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$:

$$1 - \partial_t \psi^\varepsilon - v^\varepsilon \cdot \nabla_x \psi^\varepsilon = 1 - \partial_t \varphi^\varepsilon - v^\varepsilon \cdot \nabla_x \varphi^\varepsilon = \int_V M(v') e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}}\,\mathrm{d}v'.$$

From the maximum property of $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$, the last inequality implies at this point:

$$1 - \partial_t \psi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon) - v^\varepsilon \cdot \nabla_x \psi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon) \geqslant \int_V M(v') e^{\eta(t^\varepsilon, x^\varepsilon, v^\varepsilon) - \eta(t^\varepsilon, x^\varepsilon, v')}\,\mathrm{d}v'.$$

Passing to the limit, we obtain at (t^0, x^0):

$$1 - \partial_t \psi^0(t^0, x^0) - v^* \cdot \nabla_x \psi^0(t^0, x^0) \geqslant \int_V M(v') e^{\eta(t^0, x^0, v^*) - \eta(t^0, x^0, v')}\,\mathrm{d}v' = e^{\eta(t^0, x^0, v^*)}.$$

From the very definition of the corrector η (16), this writes also

$$\forall v \in V, \quad 1 - \partial_t \psi^0(t^0, x^0) - v \cdot \nabla_x \psi^0(t^0, x^0) \geqslant e^{\eta(t^0, x^0, v)}.$$

Therefore we obtain at point (t^0, x^0),

$$\int_V \frac{M(v)}{1 - \partial_t \psi^0(t^0, x^0) - v \cdot \nabla_x \psi^0(t^0, x^0)}\,\mathrm{d}v \leqslant \int_V M(v) e^{-\eta(t^0, x^0, v)}\,\mathrm{d}v = 1.$$

We conclude that ψ^0 is a subsolution of (6). □

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