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Differential Geometry

Rigidity of automorphism groups of invariant domains in certain Stein homogeneous manifolds

Rigidité de groupe d'automorphismes d'un domaine invariant dans les variétés de Stein homogène

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ABSTRACT

Given a Stein manifold $X_{\mathbb{C}}$ which is homogeneous under a complex reductive Lie group $G_{\mathbb{C}}$, i.e., a complexification $G_{\mathbb{C}}/K_{\mathbb{C}}$ of a compact homogeneous space G/K. Consider a relatively compact domain D which is invariant w.r.t. the compact real form G of the complex reductive Lie group in the Stein manifold $X_{\mathbb{C}}$. We find a relation between the automorphism group of the invariant domain D and isometric group of the compact homogeneous space G/K. When the compact homogeneous space G/K is isotropy irreducible, or even more general, we obtain a rigidity property of the automorphism groups.

RÉSUMÉ

Soit $X_{\mathbb{C}}$ une variété de Stein qui est homogène sous un groupe de Lie réductif complexe $G_{\mathbb{C}}$, cést-à-dire, la complexification $G_{\mathbb{C}}/K_{\mathbb{C}}$ d'un espace homogène compact G/K. Soit D un domaine relativement compact qui est invariant par rapport à la forme compacte G de groupe de Lie réductif complexe dans $X_{\mathbb{C}}$. On trouve une relation entre le groupe d'automorphismes du domaine invariant D et le groupe d'isométrie de l'espace homogène compact G/K. Si l'espace homogène compact G/K est isotropie irréductible, on obtient une propriété de rigidité du groupe d'automorphismes.

1. Introduction

It is well-known in complex analysis (see [24]) that for an annulus in the complex plane, its automorphism group is just $T \rtimes Z_2$, where T is the circle group. In several complex variables, the analogue of the annulus is the relatively compact Reinhardt domain in $(\mathbb{C}^*)^n$. It is also well-known that for a Reinhardt domain $D \subset\subset (\mathbb{C}^*)^n$, the identity component $Aut(D)^0$ of the automorphism group $Aut(D)$ of D is exactly T^n, the n-dimensional torus (n-times product $T \times \cdots \times T$ of T). This result was established in 1980's in several papers, see [2,4,16,25].

In the setting of the group actions, the annulus is a relatively compact T-invariant domain in $\mathbb{C}^*$, the punctured complex plane with the usual multiplication as the group structure, which is the complexification of the group T; while the above

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Reinhardt domain is a relatively compact T^n-invariant domain in $(\mathbb{C}^*)^n$, which is the complexification of T^n. The action here is the natural action of T^n on $(\mathbb{C}^*)^n$ given by coordinate-wise multiplication.

Let G be a connected compact Lie group and K be a closed subgroup of G, then $X = G/K$ is a compact homogeneous space and $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ is a complexification of X, where $G_{\mathbb{C}}$, $K_{\mathbb{C}}$ are the universal complexifications of G, K. It is known that $X_{\mathbb{C}}$ is a Stein manifold and there is a natural holomorphic action of $G_{\mathbb{C}}$ on $X_{\mathbb{C}}$ given by the left translations. Let $D \subset X_{\mathbb{C}}$ be a G-invariant domain, in the present Note, we will consider the holomorphic automorphism group $Aut(D)$ of D. Throughout this paper, a domain means a connected open set. In [30], Zhou proved the following compactness property of the automorphism group for the relatively compact invariant domain D in $G_{\mathbb{C}}/K_{\mathbb{C}}$:

Theorem 1.1. *(See [30].) Let G be a connected compact Lie group, K a closed subgroup of G, $D \subset\subset G_{\mathbb{C}}/K_{\mathbb{C}}$ a G-invariant domain, then $Aut(D)$ is compact.*

The above result was earlier proved by G. Fels and L. Geatti under more restrictive assumption that (G, K) is a Riemannian symmetric pair [10]. It is easy to see that the automorphism group $Aut(D)$ of D obviously contains G, if the action is effective. A natural question is when there are not additional positive dimensional symmetries, i.e., when $Aut(D)/G$ is discrete. The main results in the present Note are motivated by the question and the known results about the rigidity of automorphism groups for Reinhardt domains. We deal with the question from the point of view of group actions [31]. Useful references are [8], [3], [20], [21], etc. In the present paper, we shall give a relation between the automorphism group of the invariant domains D and isometric group of the compact homogeneous space G/K. When the compact homogeneous space G/K is isotropy irreducible (except a couple of cases), or even more general, we obtain that the identity component $Aut(D)^0$ of $Aut(D)$ is just G.

2. Rigidity of automorphism groups of relatively compact invariant domains

Now let G be a general connected compact Lie group and K a closed subgroup of G, let $X = G/K$ and $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. For a G-invariant domain D in $X_{\mathbb{C}}$, one has a homomorphism of G to $Aut(D)$, we denote by $\check{G}$ the image of G in $Aut(D)$, in general, $\check{G}$ can be identified with a quotient group of G. If the action is effective, $\check{G}$ is isomorphic to G. Without loss of generality, we assume that the action is effective later on. We have the following results:

Theorem 2.1. *Let G, K, D be as above. Let D be relatively compact and orbit connected. Then the identity component $Aut(D)^0$ of $Aut(D)$ can be realized as a closed subgroup of $Iso(X, g)$, where g is some G-invariant Riemannian metric on X.*

Proof. By the main Theorem 2.6 of Zhou in [29], the envelope of holomorphy $E(D)$ of D is schlicht, i.e., $E(D) \subset X_{\mathbb{C}}$. It is clear that $E(D)$ is also relatively compact in $X_{\mathbb{C}}$. Since $G \subset Aut(D) \subset Aut(E(D))$, one may assume that D is Stein without loss of generality. Therefore D is orbit convex (see [29]). By a theorem of Zhou in [30] (p. 1109), D contains a minimal orbit, and $Aut(E(D))$ fixes the minimal orbit X and is compact. So one may find an $Aut(E(D))$-invariant Riemannian metric g on X which is certainly G-invariant, and hence there is a homomorphism $Aut(E(D)) \to Iso(X, g)$. Since X is maximal totally real in D, using the identity theorem, one has $Aut(E(D)) \subset Iso(X, g)$. □

Remark 2.1. As a consequence, one can get an immediate proof of the result on Reinhardt domains mentioned in the introduction, since the identity component of the isometric group of T^n with any T^n-invariant Riemannian metric is just T^n.

Remark 2.2. Any G-invariant domain in $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ is automatically orbit connected, if either (G, K) is a symmetric pair [17,1], or K is connected [7] (for the case when K is trivial), [29].

As a corollary, when (G, K) is a Riemannian symmetric pair, then $Aut(D)^0 = G$. The corollary is a generalization of the main result Theorem 0.1 in [10], the method used here is different from theirs.

With the above theorem, we may actually generalize the above corollary to the more general case when G/K is an isotropy irreducible space.

Definition 2.1. The space $X = G/K$ is called isotropy irreducible (or strongly isotropy irreducible) if the isotropy representation of G_x (or G_x^0) at each point $x \in X$ is irreducible, where G_x is the isotropy subgroup of G at x whose identity component is G_x^0.

An irreducible symmetric space is a special case of isotropy irreducible spaces. The G-invariant Riemannian metric on the isotropy irreducible space G/K is unique up to a constant factor.

Theorem 2.2. *Let $X = G/K$ be a strongly isotropy irreducible space, where G is a connected compact Lie group and K is a closed subgroup of G. Let $D \subset\subset G_{\mathbb{C}}/K_{\mathbb{C}}$ be a G-invariant domain. Assume that both exceptional groups G_2 and $Spin(7)$ are not the universal covering groups of G. Then $Aut(D)^0 = G$.*

Proof. One may assume that G is simply connected, since one may take the universal covering group if necessary. It is known that there is a subgroup K' of K consisting of some connected components of K such that there is a G-equivariant embedding $l: D \to G_{\mathbb{C}}/K'_{\mathbb{C}}$ such that $l(D)$ is orbit connected and $eK'_{\mathbb{C}} \in l(D)$ (cf. [14]). Since G/K' is still strongly isotropy irreducible, replacing D by $l(D)$, we may assume that D is orbit connected. By the above Theorem 2.1, one has $G \subset Aut(D) \subset Iso(X, g)$. Consider the natural projection $\pi : \tilde{X} = G/K^0 \to X = G/K$, where K^0 is the identity component of K. Since $\tilde{X}$ is still strongly isotropy irreducible, $Iso(\tilde{X}, \tilde{g})^0 = G$ [28,27]. Therefore, by the following lemma [9], $Aut(D)^0 = G$. □

Lemma 2.3. *Let M be a complex manifold such that its fundamental group $\pi_1(M)$ is finitely generated. Let $p : \tilde{M} \to M$ be a holomorphic covering of finite degree. Then the identity component $Aut(M)^0$ of the holomorphic automorphism group of M is naturally isomorphic to a quotient group of some closed subgroup of $Aut(\tilde{M})$; in particular, $\dim Aut(M) \leqslant \dim Aut(\tilde{M})$ if $\dim Aut(\tilde{M})$ is finite.*

Theorem 2.4. *Let $X = G/K$ be an isotropy irreducible homogeneous space which is not strongly isotropy irreducible, where G is a connected compact Lie group, K is a closed subgroup with $\dim K > 0$. Let $D \subset\subset G_{\mathbb{C}}/K_{\mathbb{C}}$ be a G-invariant domain. Then $Aut(D)^0 = G$.*

For the special cases $G/K = G_2/SU(3) = S^6$ and $G/K = Spin(7)/G_2 = S^7$ in Theorem 2.2 which can be also written as $SO(7)/SO(6)$ and $SO(8)/SO(7)$ respectively; in the case of S^6, if D is G_2-invariant but not $SO(7)$-invariant, then $Aut(D)^0 = G_2$; if D is not only G_2-invariant but also $SO(7)$-invariant, then $Aut(D)^0 = SO(7)$; for D in the case of S^7, one has similar consequence. The exceptional case when $\dim K = 0$ in Theorem 2.4 can be reduced to the corollary following Theorem 2.1.

3. Rigidity of automorphism groups of hyperbolic invariant domains

For a hyperbolic (in the sense of Kobayashi) Reinhardt domain D in $(\mathbb{C}^*)^n$, it is shown that the identity component of $Aut(D)$ is just the n-dimensional torus T^n [16]. It is also natural to consider the similar rigidity property for the identity component of the automorphism groups of hyperbolic invariant domains in $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. In the present Note, we establish the following results for hyperbolic invariant domains in complexifications of isotropy irreducible spaces:

Theorem 3.1. *Let $X = G/K$ be a strongly isotropy irreducible space, where G is a connected compact Lie group and K is a closed subgroup of G. Let $D \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ be a hyperbolic G-invariant domain. Assume that both G_2 and $Spin(7)$ are not the universal covering group of G. Then $Aut(D)^0 = G$.*

Proof. Using the same argument as in the proof of Theorem 2.2 and by Lemma 2.3, we may further assume that K is connected and X is simply connected.

Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{k} = Lie(K)$. The adjoint representation of G on $\mathfrak{g}$ induces a representation of K on $\mathfrak{g}$. Take a K-submodule $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then the representation of K on $\mathfrak{p}$ is irreducible since G/K is a strongly isotropy irreducible space. One can naturally identify $\mathfrak{p}$ with $T_{x_0}(G/K)$ and $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p} \oplus i\mathfrak{p}$ with $T_{x_0}D$, where $x_0 = e.K_{\mathbb{C}}$. Let $O = Aut(D)^0 x_0$ be the orbit of $Aut(D)^0$ containing the point x_0, we want to prove that $O = G/K$. If it is not the case, then one has $\dim O > \dim G/K$. Then $T_{x_0}O$ is a real K-submodule of $\mathfrak{p}_{\mathbb{C}}$ which contains $\mathfrak{p}$ strictly, hence $T_{x_0}O = \mathfrak{p}_{\mathbb{C}}$ since the action of K on $\mathfrak{p}$ is irreducible. This implies $\dim O = \dim D$, i.e., O is an open orbit in D. On the other hand, $Aut(D)$ acts properly on D as a Lie transformation group since D is hyperbolic (cf. [15]), so the orbit O is also closed in D. Hence $O = D$ since D is connected and so D is homogeneous. This implies that D is biholomorphic to a homogeneous bounded domain (cf. [22]), and hence is Stein and contractible (cf. [26]). Since K is connected, the Steinness of D implies orbit convexity of D (cf. [29]), hence it is homotopic to X. We get a contradiction since X is not contractible.

Note that $O = G/K$ is a totally real submanifold of D of maximal dimension, so any element in $Aut(D)^0$ is uniquely determined by its operation on G/K. Since $Aut(D)^0$ acts on D properly and X is compact, we see $Aut(D)^0$ is compact. Hence there is an $Aut(D)^0$-invariant Riemannian metric g on G/K which makes G/K a Riemannian strongly isotropy irreducible space. So in this case, the theorem holds again by the results in [28] on isometry groups of simply connected strongly isotropy irreducible spaces. □

Theorem 3.2. *Let $X = G/K$ be an isotropy irreducible homogeneous space which is not strongly isotropy irreducible, where G is a connected compact Lie group, K is a closed subgroup with $\dim K > 0$. Let $D \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ be a Stein G-invariant domain which is hyperbolic in the sense of Kobayashi. Then $Aut(D)^0 = G$.*

The proof of the theorem will be given elsewhere [9]. The special cases $G_2, Spin(7)$ in Theorem 3.1 and $\dim K = 0$ in Theorem 3.2 can be treated like in the last paragraph of the last section.

4. Further remarks

The above results about the case of isotropy irreducible homogeneous spaces could be extended to a more general case of the so-called “asystatic” transitive actions. The isometric groups of the asystatic spaces have also rigidity property except few cases (see [12]).

There is a relation between our result and the well-known result on the rigidity of Grauert tubes of a Riemannian symmetric space of compact type. When $M = G/K$, it is known that there is a G-equivariant biholomorphism between TM and $G_{\mathbb{C}}/K_{\mathbb{C}}$, and $Aut(T^rM)^0 = G$, see [23,5,6,13,18], etc. In this case, the connectedness of the full automorphism groups is equivalent to the rigidity of the Grauert tube. As for the connectedness of the full automorphism groups in our case, we may prove that when D is relatively compact and strictly pseudoconvex domain with smooth boundary (or more general pseudoconvex domains satisfying condition R) in $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ with dimension $\geqslant 2$, then $Aut(D)$ is connected. Therefore, $Aut(D)$ is just G when X is isotropy irreducible. The proof of the result will be given elsewhere.

In the proof of the Theorem 3.2 we need the assumption of the Steinness of the domain. The envelope of holomorphy of a hyperbolic manifold may not be hyperbolic again in general even for invariant domains. For example, there is a hyperbolic tube domain in $\mathbb{C}^2$ whose envelope of holomorphy is the whole $\mathbb{C}^2$ [19]; recently a family of hyperbolic $SU(1,1)$-invariant domains in $SL(2,\mathbb{C})/U(1)^{\mathbb{C}}$ whose envelopes of holomorphy are not hyperbolic was also constructed [11]. These domains are invariant w.r.t. to noncompact groups actions. Even for compact group action, there exits a hyperbolic Reinhardt domain in $(\mathbb{C}^*)^2$ whose envelope of holomorphy is not hyperbolic [9].

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