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Complex Analysis/Analytic Geometry

On the image of an algebraic projective space ☆

Sur l'image d'un espace algébrique projectif

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A R T I C L E   I N F O

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A B S T R A C T

We prove that if X is a projective algebraic space, Y is a normal compact complex space and $p : X \to Y$ is a surjective morphism with equidimensional fibers then Y is also projective algebraic.

R É S U M É

On démontre le résultat suivant : si X est un espace algébrique projectif, Y est un espace complexe compact normal et $p : X \to Y$ une application holomorphe surjective avec fibres équidimensionnelles alors Y est aussi un espace algébrique projectif.

1. Introduction

If X is a Kähler manifold, Y is a complex manifold, $p : X \to Y$ is a proper holomorphic map, and p is equidimensional, then it follows by a result of Varouchas [10] that Y is also Kähler. This result was extended in [11] to complex spaces with singularities (under the flatness assumption of p). On the other hand, it is known that the image of a Moishezon space by a holomorphic map is itself Moishezon (see, for example [1]). The well-known result of Moishezon [7] asserts that a compact complex manifold is projective if and only if it is Kähler and Moishezon. A similar result does not hold for spaces with singularities even for normal complex surfaces (see, e.g. [3]). In this Note we consider morphisms $p : X \to Y$ of compact complex spaces with equidimensional fibers such that X is a projective algebraic space. If X and Y are assumed to be smooth it follows from the results mentioned above that Y is projective algebraic.

Our main result (Theorem 2) states that if X is a projective algebraic space, Y is a normal compact complex space and $p : X \to Y$ is a morphism with equidimensional fibers then Y is also projective algebraic. When the fibers of p are 0-dimensional (i.e. p is a ramified covering map) this result was obtained by Remmert and Van de Ven in [8]. For fibers of positive dimension Theorem 2 was proved by C. Horst in [6], under the additional assumption that Y has isolated singularities, using an analytic version of Chevalley's criterion of projectivity (see [2]). Note that the normality assumption is essential (see e.g. [4], p. 171, Ex. 7.13 and [6]).

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2. The results

We denote by $\mathbb{P}_{\nu,n}$ the projective space that parametrizes homogeneous polynomials $F \in \mathbb{C}[z_0,\ldots,z_n]$ of degree ν. For $F \in \mathbb{P}_{\nu,n}$ we set $Z(F) := \{[z_0 : \cdots : z_n] \in \mathbb{P}^n\colon\ F(z_0,\ldots,z_n) = 0\}$.

Lemma 1. *Given a subvariety C, $\dim C \geqslant 1$, of $\mathbb{P}^n$ then $\{F \in \mathbb{P}_{\nu,n}\colon \dim(Z(F)\cap C) = \dim(C)\}$ is a finite union of linear subspaces of $\mathbb{P}_{\nu,n}$ of codimension at least $\nu+1$.*

Proof. Let C_j, $j = 1,\ldots,k$, be the irreducible components of C with $\dim C_j = \dim C$. Then we have

$$\bigl\{F \in \mathbb{P}_{\nu,n}\colon \dim\bigl(Z(F)\cap C\bigr) = \dim(C)\bigr\} = \bigcup_{j=1}^{k}\bigl\{F \in \mathbb{P}_{\nu,n}\colon\ Z(F) \supset C_j\bigr\}.$$

Obviously each $\{F \in \mathbb{P}_{\nu,n}\colon\ Z(F) \supset C_j\}$ is a linear subspace of $\mathbb{P}_{\nu,n}$.

For the codimension inequality notice that if $A_1,\ldots,A_{\nu+1}$ are distinct points on C_j for any fixed j then $\{F \in \mathbb{P}_{\nu,n}\colon F(A_l) = 0,\ l = \overline{1,\nu+1}\}$ has codimension $\nu+1$ in $\mathbb{P}_{\nu,n}$. Indeed, as $F(A_l) = 0$ is just a linear equation in $\mathbb{P}_{\nu,n}$, it suffices to show that for each $k \leqslant \nu$ there exists a homogeneous polynomial, F, of degree ν such that $F(A_1) = 0,\ldots,F(A_k) = 0$ and $F(A_{k+1}) \neq 0$. To see this let G_l be homogeneous polynomials of degree 1 in $z_0,\ldots,z_n$, $l = 1,\ldots,k$, such that $G_l(A_l) = 0$ and $G_l(A_i) \neq 0$ for $i \neq l$, $i = 1,\ldots,k+1$. We set $F = G_1^{i_1}\cdots G_k^{i_k}$ where $i_1,\ldots,i_k \geqslant 1$ are integers such that $i_1+\cdots+i_k = \nu$. □

We denote by $C_{n,k,d}$ the Chow variety that parametrizes subvarieties of degree d and dimension k of $\mathbb{P}^n$. We have then that $C_{n,k,d}$ is a quasi-projective variety and that the incidence set $\{(X,z) \in C_{n,k,d} \times \mathbb{P}^n\colon\ z \in X\}$ is an algebraic subset of $C_{n,k,d} \times \mathbb{P}^n$ (see, for example [9]).

Theorem 1. *Suppose that n, k and d are integers, $n,k,d \geqslant 1$. Then there exists $\nu_0 \in \mathbb{Z}$, $\nu_0 \geqslant 1$, such that for every $\nu \in \mathbb{Z}$, $\nu \geqslant \nu_0$ there exists a homogeneous polynomial $F \in \mathbb{C}[z_0,z_1,\ldots,z_n]$ of degree ν with the property that $Z(F) \subset \mathbb{P}^n$ contains no subvariety of $\mathbb{P}^n$ of dimension k and degree at most d.*

Proof. For $1 \leqslant j \leqslant d$ let

$$H_j := \bigl\{(X,F) \in C_{n,k,j} \times \mathbb{P}_{\nu,n}\colon \dim\bigl(Z(F)\cap X\bigr) = k\bigr\}.$$

We prove that H_j is a closed algebraic subset of $C_{n,k,j} \times \mathbb{P}_{\nu,n}$. Let

$$\tilde{H}_j = \bigl\{(X,z,F) \in C_{n,k,j} \times \mathbb{P}^n \times \mathbb{P}_{\nu,n}\colon\ z \in X,\ F(z) = 0\bigr\}.$$

Notice that $\tilde{H}_j = \tilde{H}'_j \cap \tilde{H}''_j$ where

$$\tilde{H}'_j = \bigl\{(X,z) \in C_{n,k,j} \times \mathbb{P}^n\colon\ z \in X\bigr\} \times \mathbb{P}_{\nu,n}$$

and

$$\tilde{H}''_j = C_{n,k,j} \times \bigl\{(z,F) \in \mathbb{P}^n \times \mathbb{P}_{\nu,n}\colon\ F(z) = 0\bigr\}.$$

As both $\tilde{H}'_j$ and $\tilde{H}''_j$ are closed algebraic subsets of $C_{n,k,j} \times \mathbb{P}^n \times \mathbb{P}_{\nu,n}$ it follows that $\tilde{H}_j$ is a closed algebraic subset of $C_{n,k,j} \times \mathbb{P}^n \times \mathbb{P}_{\nu,n}$. Let $\pi_j : C_{n,k,j} \times \mathbb{P}^n \times \mathbb{P}_{\nu,n} \to C_{n,k,j} \times \mathbb{P}_{\nu,n}$ be the canonical projection. As $\mathbb{P}^n$ is compact it follows that π_j is proper. If we denote by $\tilde{\pi}_j : \tilde{H}_j \to C_{n,k,j} \times \mathbb{P}_{\nu,n}$ the restriction of π_j to $\tilde{H}_j$ we have that $\tilde{\pi}_j$ is also proper. It follows that $\{(X,F) \in C_{n,k,j} \times \mathbb{P}_{\nu,n}\colon\ \dim \tilde{\pi}_j^{-1}(X,F) \geqslant k\}$ is an analytic subset of $C_{n,k,j} \times \mathbb{P}_{\nu,n}$ (by the semi-continuity of the dimension of the fibers in the Zariski topology, see e.g. [12], p. 240). However $\pi_j^{-1}(X,F) = \{X\} \times (Z(F)\cap X) \times \{F\}$ and therefore $\dim \tilde{\pi}_j^{-1}(X,F) \leqslant k$. We deduce that $\{(X,F) \in C_{n,k,j} \times \mathbb{P}_{\nu,n}\colon\ \dim \tilde{\pi}_j^{-1}(X,F) \geqslant k\} = H_j$ and hence H_j is a closed analytic subset of $C_{n,k,j} \times \mathbb{P}_{\nu,n}$ as claimed.

Let $p_{1,j} : H_j \to C_{n,k,j}$, $p_{2,j} : H_j \to \mathbb{P}_{\nu,n}$ be the canonical projections. From Lemma 1 it follows that the fibers of $p_{1,j}$ have dimension at most $\dim(\mathbb{P}_{\nu,n}) - \nu - 1$ and therefore $\dim(H_j) \leqslant \dim C_{n,k,j} + \dim(\mathbb{P}_{\nu,n}) - \nu - 1$. If we choose $\nu \geqslant \max\{\dim C_{n,k,j}\colon\ j = 1,\ldots,d\}$ then $\dim(H_j) < \dim \mathbb{P}_{\nu,n}$. Here the projections p_{2j} are not necessarily proper, but we can nevertheless conclude that $\bigcup_{j=1}^{d} p_{2,j}(H_j)$ is of Hausdorff dimension $\leqslant 2n-2$ in $\mathbb{P}_{\nu,n}$. Therefore for almost every polynomial $F \in \mathbb{P}_{\nu,n}$ we will have that $Z(F)$ does not contain any irreducible component of X of dimension k for any $X \in C_{n,k,j}$, $j \leqslant d$. □

For the next lemma we assume that X is a closed subvariety of $\mathbb{P}^n$, Y is a reduced compact complex space and $p : X \to Y$ is a surjective morphism. For $y \in Y$ we set $X_y := p^{-1}(y)$. If $\dim X_y = m$ we denote by $X_y^{(m)}$ the union of all irreducible components of X_y of dimension m.

Lemma 2. *If the fibers of p, X_y, $y \in Y$, have all the same dimension m then there exists an integer d such that* $\deg X_y^{(m)} \leqslant d$ *for every* $y \in Y$.

Proof. We prove first that if $y_0 \in Y$ is any point then there exists U a neighborhood of y_0 and an integer d_U such that $\deg X_y^{(m)} \leqslant d_U$ for every $y \in U$. Indeed, let L be a linear subspace of $\mathbb{P}^n$ such that $\dim L = n - m - 1$ and $L \cap X_{y_0} = \emptyset$. For some small connected neighborhood U of y_0 we have that $L \cap X_y = \emptyset$ for every $y \in U$. Let $X(U) = p^{-1}(U)$. Note that $\mathbb{P}^n \setminus L$ has the structure of a holomorphic vector bundle $\pi : \mathbb{P}^n \setminus L \to \mathbb{P}^m$ and, for any $y \in U$, $\pi|_{X_y^{(m)}} : X_y^{(m)} \to \mathbb{P}^m$ is a branched covering of degree $d_y = \deg X_y^{(m)}$. We consider the analytic map $G : X(U) \to \mathbb{P}^m \times U$, $G(x) = (\pi(x), p(x))$. Then G is a proper finite surjective morphism. It follows that there exists an integer d_U such that $d_y \leqslant d_U$ for every $y \in U$.

The conclusion of the lemma follows now from the compactness of Y. □

Theorem 2. *Suppose that X and Y are reduced compact complex spaces and $p : X \to Y$ is a surjective holomorphic mapping. We assume that X is projective algebraic, Y is normal and the fibers of p have all the same dimension. Then Y is projective algebraic.*

Proof. We will prove the theorem by induction on the dimensions of the fibers of p. If p has discrete fibers this was proved in [8].

We assume now that we proved our theorem for every morphism such that each fiber has dimension $k - 1$, $k \geqslant 1$, and we consider a proper surjective holomorphic mapping $p : X \to Y$ such that $X_y := p^{-1}(y)$ has dimension k for each $y \in Y$. Let $X \hookrightarrow \mathbb{P}^n$ be an embedding of X. It follows from Lemma 2 that there exists a positive integer d such that $\deg X_y^{(k)}$ is at most d for every $y \in Y$. We apply then Theorem 1 and we deduce that there exists $F \in \mathbb{C}[z_0, z_1, \ldots, z_n]$, a homogeneous polynomial of sufficiently large degree, such that $Z(F)$ contains no irreducible component of dimension k of the fibers of p. Then for every $y \in Y$ we have that $Z(F) \cap X_y \neq \emptyset$ and $\dim Z(F) \cap X_y = k - 1$. If we let $X_1 := Z(F) \cap X$ and $p_1 : X_1 \to Y$ to the restriction of p we can apply the induction hypothesis and deduce that Y is projective algebraic. □

Remark. In [5], Theorem 2.6, Hironaka studied proper mappings $\pi : X \to Y$ in the algebraic category and showed that a generic hypersection of X does not contain any fiber of Y. However his arguments rely heavily on the algebraicity of Y.

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References

[1] V. Ancona, G. Tomassini, Modifications Analytiques, Lecture Notes in Mathematics, vol. 943, Springer-Verlag, Berlin–New York, 1982.
[2] G. Fischer, Projektive Einbettung komplexer Mannigfaltigkeiten, Math. Ann. 234 (1) (1978) 45–50.
[3] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann. 146 (1962) 331–368.
[4] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York–Heidelberg, 1977.
[5] H. Hironaka, Smoothing of algebraic cycles of small dimensions, Amer. J. Math. 90 (1968) 1–54.
[6] C. Horst, Über Bilder projektiv-algebraischer Räume, J. Reine Angew. Math. 324 (1981) 136–140.
[7] B.G. Moĭšezon, On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions, Amer. Math. Soc. Transl. 63 (1967) 51–177.
[8] R. Remmert, T. van de Ven, Über holomorphe Abbildungen projektiv-algebraischer Mannigfaltigkeiten auf komplexe Räume, Math. Ann. 142 (1961) 453–486.
[9] I.R. Shafarevich, Basic Algebraic Geometry, Die Grundlehren der Mathematischen Wissenschaften, Band 213, Springer-Verlag, New York–Heidelberg, 1974.
[10] J. Varouchas, Stabilité de la classe des variétés kählériennes par certains morphismes propres, Invent. Math. 77 (1) (1984) 117–127.
[11] J. Varouchas, Kähler spaces and proper open morphisms, Math. Ann. 283 (1) (1989) 13–52.
[12] H. Whitney, Complex Analytic Varieties, Addison–Wesley Publishing Co., Reading, MA, 1972.