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Partial Differential Equations/Differential Geometry

# A Note on the Bernstein property of a fourth order complex partial differential equations

## *Sur la propriété de Bernstein des équations différentielles partielles complexes d'ordre quatre*

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A B S T R A C T

For a smooth strictly plurisubharmonic function $u$ on an open set $\Omega \subset \mathbb{C}^n$ and $F$ a $C^1$ nondecreasing function on $\mathbf{R}_+^*$, we investigate the complex partial differential equations

$$\Delta_g \log\det(u_{i\bar{j}}) = F\big(\det(u_{i\bar{j}})\big)\big\|\nabla_g \log\det(u_{i\bar{j}})\big\|_g^2,$$

where $\Delta_g$, $\|.\|_g$ and $\nabla_g$ are the Laplacian, tensor norm and the Levi-Civita connexion, respectively, with respect to the Kähler metric $g = \partial\bar{\partial}u$. We show that the above PDE's has a Bernstein property, i.e. $\det(u_{i\bar{j}})$ is constant on $\Omega$, provided that $g$ is complete, the Ricci curvature of $g$ is bounded below and $F$ satisfies $\inf_{t\in\mathbf{R}^+}(2tF'(t) + \frac{F(t)^2}{n}) > \frac{1}{4}$ and $F(\max_{B(R)} \det u_{i\bar{j}}) = o(R)$.



R É S U M É

Pour une fonction $u$ strictement plurisouharmonique de classe $C^\infty$ sur un ouvert $\Omega$ de $\mathbb{C}^n$ et $F$ une fonction de classe $C^1$ croissante sur $\mathbb{R}^+$, on considère l'équation aux dérivées partielles complexes

$$\Delta_g \log\det(u_{i\bar{j}}) = F\big(\det(u_{i\bar{j}})\big)\big\|\nabla_g \log\det(u_{i\bar{j}})\big\|_g^2,$$

où $\Delta_g$, $\|.\|_g$ et $\nabla_g$ sont respectivement le Laplacian, la norme et la connexion de Levi-Civita par rapport à la métrique Kählerienne $g = \partial\bar{\partial}u$. On montre que l'EDP précédente vérifie la propriété de Bernstein, i.e. $\det(u_{i\bar{j}})$ est constante sur $\Omega$, pourvu que $g$ soit complète, la courbure de Ricci de $g$ soit minorée et $F$ satisfasse $\inf_{t\in\mathbf{R}^+}(2tF'(t) + \frac{F(t)^2}{n}) > \frac{1}{4}$ et $F(\max_{B(R)} \det u_{i\bar{j}}) = o(R)$.



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## 1. Introduction

In this Note, we investigate a class of fourth order complex nonlinear partial differential equations for a strictly plurisubharmonic function $u$ on an open set $\Omega \subset \mathbb{C}^n$:

$$\Delta_g \log\det(u_{i\bar{j}}) = F\big(\det(u_{i\bar{j}})\big)\big\|\nabla_g \log\det(u_{i\bar{j}})\big\|_g^2 \tag{1}$$

where $\Delta_g$, $\|.\|_g$ and $\nabla_g$ are the Laplacian, tensor norm and the Levi-Civita connexion, respectively, with respect to the Kähler metric $g = \partial\bar{\partial}u$. For $n \geqslant 2$, Eq. (1) is the Euler–Lagrange equation of the variational problem of the functional

$$\mathcal{V}_\Phi(u) = \int_\Omega \Phi\big(\det(u_{i\bar{j}})\big)\,\mathrm{d}V_e \tag{2}$$

for some $\Phi \in C^4(\mathbf{R}_+^*)$. If $\Phi(t) = t^{1-\beta}$, $\beta \neq 1$, the Euler–Lagrange equation of $\mathcal{V}_\Phi$ is

$$\Delta_g \log\det(u_{i\bar{j}}) = \beta\big\|\nabla_g \log\det(u_{i\bar{j}})\big\|_g^2 \tag{3}$$

In [1], Chen and Li showed the following Bernstein property of the solution of (3).

**Theorem 1.1.** *Let $\Omega \subset \mathbb{C}^n$ and $u \in C^2(\Omega)$ a strictly plurisubharmonic solution of* (3)*. Assume that*

(i) $(\Omega, g)$ *is complete,*
(ii) $Ric(g) \geqslant -C_1$ $(K \geqslant 0)$,
(iii) $\det(u_{i\bar{j}}) < C_2$

*where $C_1, C_2 > 0$ are real constants. If $\beta > \frac{\sqrt{9n^2+4n}+3n}{4}$ then $\det(u_{i\bar{j}})$ is constant on $\Omega$.*

For more details about Bernstein property, one can see the paper [1] and references therein. In this Note, we will show the Bernstein property for the solutions of Theorem 1.1. Our main result is the following:

**Theorem 1.2.** *Let $\Omega \subset \mathbb{C}^n$ be an open set and $u \in C^2(\Omega)$ a strictly plurisubharmonic solution of* (1)*. Assume that*

(i) $(\Omega, g)$ *is complete,*
(ii) $Ric(g) \geqslant -K$ $(K \geqslant 0)$,
(iii) $F \in C^1(]0, +\infty[)$ *nondecreasing satisfying*

$$\inf_{t>0}\left(2tF'(t) + \frac{F(t)^2}{n}\right) > \frac{1}{4} \quad \text{and} \quad F\Big(\max_{B(R)}\det(u_{i\bar{j}})\Big) = o(R) \tag{4}$$

*where $B(R) = \{z \in \Omega, d_g(z, z_0) < R\}$. Then $\det(u_{i\bar{j}})$ is constant on $\Omega$.*

If $F(t) = \beta$, the following corollary extends Theorem 1.1.

**Corollary 1.3.** *Let $\Omega \subset \mathbb{C}^n$ be an open set and $u \in C^2(\Omega)$ a strictly plurisubharmonic solution of* (3)*. Assume that $(\Omega, g)$ is complete and $Ric(g) \geqslant -K$ $(K \geqslant 0)$. If $|\beta| > \frac{\sqrt{n}}{2}$ then $\det(u_{i\bar{j}})$ is constant on $\Omega$.*

**Examples.** Examples of function $F$ which satisfies conditions of Theorem 1.2:

1. $F(t) = \log^{2\alpha} t$ where $\alpha \in ]\alpha_0(n), \frac{1}{2}]$, $\alpha_0(n) > 0$.
2. $F(t) = (\log t)\log(\alpha + (\log t)^2)$ where $\alpha > e^{\frac{1}{4}}$.
3. $F(t) = f(\log t)$ where $f$ is a solution of the Riccati equation $2y' + \frac{y^2}{n} = \alpha$, $\alpha > \frac{1}{4}$.

## 2. Proof of Theorem 1.2

**Proof.** We denote $\Psi = |\nabla U|^2$ where $U = \log\det(u_{i\bar{j}})$ and $f(t) = F(e^t)$. By Bochner formulae

$$\begin{aligned}\Delta_g\Psi &= 2f'(U)\Psi^2 + f(U)\big(\langle\nabla_g\Psi, \nabla_g U\rangle + \langle\nabla_g U, \nabla_g\Psi\rangle\big)\\ &\quad + \sum u^{i\bar{j}}u^{k\bar{l}}U_{i\bar{l}}U_{\bar{j}l} + \sum u^{i\bar{j}}u^{k\bar{l}}U_{\bar{i}l}U_{\bar{j}k} + \sum u^{i\bar{j}}u^{m\bar{s}}U_{i\bar{l}}U_m U_{\bar{j}}R_{i\bar{s}}\end{aligned} \tag{5}$$

where $(u^{i\bar{j}})$ is the inverse matrix of $(u_{i\bar{j}})$ and $R_{i\bar{s}}$ is the Ricci curvature of $g$

$$R_{i\bar{s}} = -\frac{\partial^2 \log\det(u_{k\bar{l}})}{\partial z_i \partial \bar{z}_s} = \frac{\partial^2 \log U}{\partial z_i \partial \bar{z}_s}$$

Let $z \in \Omega$ fixed, we can choose normal coordinates $(z_1, \dots, z_n)$ at $z$ such that

$$u_{i\bar{j}} = \delta_{ij}, \qquad U_1 = U_{\bar{1}}, \qquad U_i = U_{\bar{i}} = 0, \quad \forall i > 1$$

At $z$ the formula (5) becomes

$$\begin{aligned} \frac{\Delta_g \Psi}{\Psi} &= 2f'(U)\Psi + \left(f(U) - \frac{1}{2}\right)\left(\frac{\Psi_1}{\Psi}U_{\bar{1}} + U_1\frac{\Psi_{\bar{1}}}{\Psi}\right) + \frac{1}{2}\left(\frac{\Psi_1}{\Psi}U_{\bar{1}} + U_1\frac{\Psi_{\bar{1}}}{\Psi}\right) \\ &\quad + \frac{\sum U_{ik}U_{\bar{j}\bar{k}}}{\Psi} + \frac{\sum U_{i\bar{k}}U_{\bar{i}k}}{\Psi} - \frac{U_{1\bar{1}}U_1U_{\bar{1}}}{\Psi} \end{aligned} \tag{6}$$

Since $\sum U_{ik}U_{\bar{i}\bar{k}} \geqslant U_{11}U_{\bar{1}\bar{1}}$ and

$$\begin{aligned} \frac{1}{2}\left(\frac{\Psi_1}{\Psi}U_{\bar{1}} + U_1\frac{\Psi_{\bar{1}}}{\Psi}\right) - \frac{U_{1\bar{1}}U_1U_{\bar{1}}}{\Psi} &= \frac{1}{2\Psi}2\Re(U_{11}U_{\bar{1}}U_{\bar{1}}) \\ &\geqslant \frac{U_{11}U_{\bar{1}\bar{1}}}{\Psi} - \frac{1}{4}\Psi \end{aligned}$$

we obtain

$$\frac{1}{2}\left(\frac{\Psi_1}{\Psi}U_{\bar{1}} + U_1\frac{\Psi_{\bar{1}}}{\Psi}\right) - \frac{U_{1\bar{1}}U_1U_{\bar{1}}}{\Psi} + \frac{\sum U_{ik}U_{\bar{i}\bar{k}}}{\Psi} \geqslant -\frac{1}{4}\Psi \tag{7}$$

Let $G$ be the function defined on $B(R) = B_g(z_0, R)$ by

$$G(z) = \left(R^2 - r(z)^2\right)^2 \Psi$$

where $r(z) = d_g(z, z_0)$. Since $G$ vanishes at the boundary $\partial B(R)$, the maximum of $G$ is attained on $w \in B(R)$. By the maximum principle

$$\nabla_g G(w) = 0$$
$$\Delta_g G(w) \leqslant 0$$

A computation at $w$ gives

$$\frac{\Psi_i}{\Psi} = \frac{4rr_i}{R^2 - r^2} \tag{8}$$

$$\begin{aligned} \Delta_g G &= 2\Psi \left\|\nabla_g\left(R^2 - r^2\right)\right\|^2 + 2\left(R^2 - r^2\right)\Psi\Delta_g\left(R^2 - r^2\right) \\ &\quad + 2\left(R^2 - r^2\right)\sum\left(R^2 - r^2\right)_i\Psi_i + \left(R^2 - r^2\right)^2\Delta_g\Psi \\ &= 2\Psi r^2\|\nabla_g r\|^2 - 2\Psi\left(R^2 - r^2\right)\Delta_g r^2 + \left(R^2 - r^2\right)^2\Delta_g\Psi \\ &\quad - 4r\left(R^2 - r^2\right)\sum_i r_i\Psi_i \leqslant 0 \end{aligned} \tag{9}$$

Since $\|\nabla_g r\|_g = 1$, (8) and (9) imply

$$\frac{\Delta_g \Psi}{\Psi} \leqslant \frac{2\Delta_g r^2}{R^2 - r^2} + \frac{14r^2}{(R^2 - r^2)^2} \tag{10}$$

Let $\eta >$ fixed such that $\inf_{t\in\mathbb{R}}(2f'(t) + \frac{f(t)^2}{n}) > 2\eta + \frac{1}{4}$. By Schwarz inequality

$$\begin{aligned} \left(f(U) - \frac{1}{2}\right)\left(\frac{\Psi_1}{\Psi}U_{\bar{1}} + U_1\frac{\Psi_{\bar{1}}}{\Psi}\right) &= 2\Re\left(U_1\frac{4rr_1(f(U) - \frac{1}{2})}{R^2 - r^2}\right) \\ &\geqslant -\eta\Psi - \frac{16(f(U) - \frac{1}{2})^2r^2}{\eta(R^2 - r^2)^2} \end{aligned} \tag{11}$$

Since $\Delta U = f(U)$ we have

$$\frac{\sum U_{i\bar{k}}U_{\bar{i}k}}{\Psi} \geqslant \frac{\sum (U_{i\bar{i}})^2}{\Psi} \geqslant \frac{(\sum U_{i\bar{i}})^2}{n\Psi} = \frac{(\Delta U)^2}{n\Psi} = \frac{(f(U))^2}{n}\Psi \tag{12}$$

The inequalities (6), (11) and (12) give

$$\frac{\Delta_g \Psi}{\Psi} \geqslant \eta\Psi - \frac{16(f(U) - \frac{1}{2})^2 r^2}{\eta(R^2 - r^2)^2} \tag{13}$$

By Laplacian comparison theorem [2]: $\Delta_g r^2 \leqslant 2n + 2(n-1)\sqrt{K}r$. Since $|f(U) - \frac{1}{2}|^2 \leqslant 2f(U)^2 + \frac{1}{2}$ and $f$ is nondecreasing, the inequalities (10) and (11) give on $B(R)$

$$\left(R^2 - r^2\right)^2 \Psi \leqslant G(w) \leqslant C_1(1+R)R^2 + C_2\Big(f\Big(\max_{B(R)} U\Big)\Big)^2 R^2$$

Hence if $z \in B(R/2)$, we have

$$\Psi(z) \leqslant C_1\frac{1+R}{R^2} + C_2\left(\frac{f(\max_{B(R)} U)}{R}\right)^2$$

Since $(\Omega, g)$ is complete, letting $R$ to infinity, we obtain $\Psi(z) = \|\nabla_g U(z)\|^2 = 0$, i.e. $\det(u_{i\bar{j}}) = C$ on $\Omega$. $\square$

## References

[1] G. Chen, L. Sheng, A Bernstein property of a class of fourth order complex partial differential equations, Results Math. 58 (2010) 81–92.
[2] R.E. Greene, H. Wu, Function Theory on Manifolds Which Possess a Pole, Lecture Notes in Math., vol. 699, 1979.