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Disjoint pairs for $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$

Paires disjointes pour $GL(n,\mathbb{R})$ et $GL(n,\mathbb{C})$

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ABSTRACT

We show the disjointness property of Klyachko for $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$.

RÉSUMÉ

Nous montrons la propriété de disjonction de Klyachko pour $GL_n(\mathbb{R})$ et $GL_n(\mathbb{C})$.

1. Introduction

A finite family of subgroups of GL_n, each endowed with a character, was introduced by Klyachko in [9]. Over a finite field this family provides a model for GL_n (see [7]). In this note we consider the archimedean case and prove pairwise disjointness of Klyachko pairs in a sense we now explain.

Definition 1.1. Let G be a real reductive group, H_i a closed subgroup and χ_i a continuous character of H_i, $i=1,2$. We say that $(G,(H_1,\chi_1))$ and $(G,(H_2,\chi_2))$ are *disjoint pairs* if for every irreducible admissible smooth Fréchet representation of moderate growth π of G (see Section 2) we have

$$\dim \operatorname{Hom}_{H_1}(\pi,\chi_1)\cdot \dim \operatorname{Hom}_{H_2}(\pi,\chi_2)=0.$$

In order to formulate our main result we introduce some notation. In Section 3 we use this notation without further mention. Let F equal either $\mathbb{R}$ or $\mathbb{C}$ and let ψ be a non-trivial unitary character of F. Set $X_n=GL_n(F)$, let U_n be the subgroup of upper uni-triangular matrices in X_n and let ψ_n be the character of U_n defined by

$$\psi_n(u)=\psi(u_{1,2}+\cdots+u_{n-1,n}),\quad u\in U_n.$$

Let $w_n=(\delta_{i,n+1-j})\in X_n$ and let

$$J_n=\begin{pmatrix}0 & w_n\\ -w_n & 0\end{pmatrix}\in X_{2n}.$$

Consider the symplectic group Sp_{2n} defined by

$$Sp_{2n}=\{g\in X_{2n}\colon\ {}^t g J_n g=J_n\}.$$

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Fix $n \in \mathbb{N}$. For $0 \leqslant r \leqslant n$ such that $n - r = 2k$ is even consider the Klyachko subgroup $H_{r,n}$ of X_n defined by

$$H_{r,n} = \left\{ \begin{pmatrix} u & x \\ 0 & h \end{pmatrix} : u \in U_r,\ x \in M_{r\times 2k}(F),\ h \in Sp_{2k} \right\}$$

and let $\psi_{r,n}$ be the character of $H_{r,n}$ defined by

$$\psi_{r,n} \begin{pmatrix} u & x \\ 0 & h \end{pmatrix} = \psi_r(u).$$

Theorem 1.1. *The pairs $(X_n, (H_{r,n}, \psi_{r,n}))$, $0 \leqslant r \leqslant n$, $r \equiv n \bmod 2$ are pairwise disjoint.*

The analogous result was obtained in [7] over a finite field and in [10] over a non-archimedean local field.

2. Generalities

Let G be a group, V a vector space over $\mathbb{C}$ and $\chi : G \to \mathbb{C}^*$ a group homomorphism. For a representation (π, V) of G on V, i.e. a group homomorphism $\pi : G \to GL(V)$, let

$$V^{G,\chi} = \{ v \in V \colon \pi(g)v = \chi(g)v \text{ for all } g \in G \}.$$

If χ is trivial we also denote $V^{G,\chi}$ by V^G. Denote by $\pi \otimes \chi$ (or sometimes by $V \otimes \chi$) the representation of G on V defined by $g \mapsto \chi(g)\pi(g)$. Note that $V^{G,\chi} = (V \otimes \chi^{-1})^G$.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. If V is a $\mathfrak{g}$-module let $V^{\mathfrak{g}} = \{v \in V \colon \mathfrak{g}v = 0\}$ be the subspace annihilated by $\mathfrak{g}$ and $V_{\mathfrak{g}} = V/\mathfrak{g}V$ the space of co-invariants.

We refer to [2] for the notions of Schwartz functions and Schwartz distributions in the following setting. For a Nash manifold X we denote by $\mathcal{S}(X)$ the Fréchet space of $\mathbb{C}$ valued Schwartz functions on X and by $\mathcal{S}^*(X)$ its topological dual, the space of Schwartz distributions.

Let G be a Nash group with a Nash action on X and let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathcal{S}(X)$ (and therefore also $\mathcal{S}^*(X)$) is naturally a $\mathfrak{g}$-module. Let χ be a character of G, i.e., a smooth group homomorphism $\chi : G \to \mathbb{C}^*$. Then $\mathcal{S}^*(X) \otimes \chi^{-1}$ is also a $\mathfrak{g}$-module and evidently

$$\mathcal{S}^*(X)^{G,\chi} \subseteq \left(\mathcal{S}^*(X) \otimes \chi^{-1}\right)^{\mathfrak{g}}. \tag{2.1}$$

For every $x \in X$ denote by Gx the G-orbit of x, by G_x the stabilizer of x in G and by $\mathfrak{g}_x$ the Lie algebra of G_x. Let $T(X)$ be the tangent bundle of X. For a Nash submanifold Y of X let $N_Y^X = (T(X)|_Y)/T(Y)$ be the normal bundle to Y in X and let $\mathrm{CN}_Y^X = (N_Y^X)^*$ be the conormal bundle. For a point $y \in Y$ we denote by $N_{Y,y}^X$ (resp. $\mathrm{CN}_{Y,y}^X$) the fiber over y in N_Y^X (resp. CN_Y^X), i.e., the normal (resp. conormal) space to Y in X at the point y.

If X is itself a Nash group and H_i is a closed subgroup, $i = 1, 2$, then we shall always consider the left action of $H_1 \times H_2$ on X defined by $((h_1, h_2), x) \mapsto h_1 x h_2^{-1}$ for $h_1 \in H_1$, $h_2 \in H_2$ and $x \in X$.

Let G be a real reductive group. An admissible smooth Fréchet representation of moderate growth π of G is a representation in the category $\mathcal{FH}(G)$ defined in [12, 11.6.8]. It is called a smooth F-representation in [4] and a Casselman–Wallach representation in [11]. We denote by $\tilde{\pi}$ the contragredient of π.

The following is an immediate consequence of [11, Theorem 2.3 (b)]. The statement in [11] is in terms of tempered generalized functions rather than Schwartz distributions. The translation is straightforward.

Theorem 2.1 *(Sun–Zhu). Let G be a real reductive group, H_i a closed subgroup and χ_i a continuous character of H_i, $i = 1, 2$. If $\mathcal{S}^*(G)^{H_1\times H_2, \chi_1^{-1}\times\chi_2^{-1}} = 0$ then for every irreducible admissible smooth Fréchet representation of moderate growth π of G we have*

$$\dim \operatorname{Hom}_{H_1}(\pi, \chi_1) \cdot \dim \operatorname{Hom}_{H_2}(\tilde{\pi}, \chi_2) = 0.$$

Next we provide a sufficient condition for vanishing of the space of equivariant distributions in an algebraic context.

Lemma 2.2. *Let $G = \mathbb{G}_a(F)$ $(= F)$ and let $\mathfrak{g}$ $(= F)$ be the Lie algebra of G. Let $\chi : G \to \mathbb{C}^*$ be a non-trivial character and let π be a finite-dimensional algebraic representation of G. Then $(\pi \otimes \chi)_{\mathfrak{g}} = 0$.*

Proof. Since π is algebraic and G unipotent, the only eigenvalue of $\pi \otimes \chi$ on G is χ. The derivative of χ at zero is not zero and therefore every non-zero element of $\mathfrak{g}$ acts on $\pi \otimes \chi$ by an invertible linear transformation. Hence $\mathfrak{g}(\pi \otimes \chi) = \pi \otimes \chi$ and there are no non-zero coinvariants. $\square$

Proposition 2.3. *Let G be an F-linear algebraic group acting on a smooth algebraic variety X. Let $\chi : G \to \mathbb{C}^*$ be a unitary character and assume that for every $x \in X$ there exists a unipotent element $u \in G_x$ such that $\chi(u) \neq 1$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.*

Proof. By (2.1) we have $T|_{\mathfrak{g}(\mathcal{S}(X)\otimes\chi^{-1})} \equiv 0$ for every $T \in \mathcal{S}^*(X)^{G,\chi}$. It is therefore enough to show that $\mathcal{S}(X)\otimes\chi^{-1} = \mathfrak{g}(\mathcal{S}(X)\otimes\chi^{-1})$. By [1, Theorem 2.2.15] it is enough to show that $(\mathrm{Sym}^k(CN^X_{Gx,x})\otimes\chi')_{\mathfrak{g}_x} = 0$ for all $k \in \mathbb{Z}_{\geqslant 0}$ where $\chi' = \chi^{-1}|_{G_x}\cdot((\Delta_G)|_{G_x}/\Delta_{G_x})$ and Δ_H denotes the modulus function of a locally compact group H.

Since χ is unitary and $(\Delta_G)|_{G_x}/\Delta_{G_x}$ positive we have $\chi'(u) \neq 1$. Since u is unipotent it lies in the image of some algebraic homomorphism $\varphi : F \to G_x$ (see e.g. [5, Proposition 5.29]). Let $\mathfrak{u}$ be the Lie algebra of $\varphi(F)$. It follows from Lemma 2.2 that $(\mathrm{Sym}^k(CN^X_{Gx,x})\otimes\chi')_{\mathfrak{u}} = 0$ and since $\mathfrak{u} \subseteq \mathfrak{g}_x$ also that $(\mathrm{Sym}^k(CN^X_{x,Gx})\otimes\chi')_{\mathfrak{g}_x} = 0$. The theorem follows. □

Remark 2.1. See [8, Lemma 3.4] for a related result.

Let ψ be a unitary character of F and G an F-linear algebraic group. A character χ of G is *ψ-algebraic* if there exists an F-algebraic homomorphism $\phi : G \to \mathbb{G}_a(F)$ such that $\chi = \psi\circ\phi$.

Corollary 2.4. *With the above notation assume that G acts on a smooth algebraic variety X. Let χ be a ψ-algebraic character of G such that $\chi|_{G_x} \not\equiv 1$ for every $x \in X$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.*

Proof. Let $\phi : G \to \mathbb{G}_a(F)$ be as above. For $x \in X$ the stabilizer G_x is an F-linear algebraic group and therefore each of its elements has a Jordan decomposition in G_x (see e.g. [6, §34.2]). If $\chi(s) \neq 1$ for some semi-simple $s \in G_x$ then let S be an F-torus in G_x containing s. Then $\phi|_S$ is a non-trivial algebraic homomorphism from a non-trivial F-torus to the additive group $\mathbb{G}_a(F)$, which is a contradiction. Thus $\chi(u) \neq 1$ for some unipotent element $u \in G_x$. The corollary therefore follows from Proposition 2.3. □

Theorem 2.5. *Let X be an F-reductive group, H_i an algebraic subgroup and χ_i a ψ-algebraic character of H_i, $i = 1, 2$. Set $G = H_1 \times H_2$ and $\chi = \chi_1 \times \chi_2$ and assume that $\chi|_{G_x} \not\equiv 1$ for all $x \in X$.*

(1) *For every irreducible admissible smooth Fréchet representation of moderate growth π of X we have*

$$\dim \mathrm{Hom}_{H_1}(\pi, \chi_1)\cdot\dim \mathrm{Hom}_{H_2}(\tilde{\pi}, \chi_2) = 0.$$

(2) *If $X = GL_n(F)$ and ι is the involution on X defined by $g^\iota = {}^tg^{-1}$ then $(X, (H_1, \chi_1))$ and $(X, (H_2^\iota, \chi_2^\iota))$ are disjoint pairs.*

Proof. The first part is immediate from Theorem 2.1 and Corollary 2.4. (Note that χ^{-1} is ψ^{-1}-algebraic and $\chi^{-1}|_{G_x} \not\equiv 1$, $x \in X$.) For $X = GL_n(F)$ it follows from [3, Theorem 2.4.2] that for every irreducible admissible smooth Fréchet representation of moderate growth π of X we have $\pi^\iota \simeq \tilde{\pi}$. Thus,

$$\mathrm{Hom}_{H_2}(\tilde{\pi}, \chi_2) \simeq \mathrm{Hom}_{H_2}\big(\pi^\iota, \chi_2\big) \simeq \mathrm{Hom}_{H_2^\iota}\big(\pi, \chi_2^\iota\big).$$

The second part therefore follows from the first. □

3. Disjointness

Fix $n \in \mathbb{N}$ and $0 \leqslant r \neq r' \leqslant n$ such that $r \equiv n \equiv r' \bmod 2$. Let ι be the involution on X_n defined by $g^\iota = {}^tg^{-1}$, $G = H_{r,n} \times H^\iota_{r',n}$ and $\theta = \psi_{r,n} \times \psi^\iota_{r',n}$ a unitary character of G. Clearly θ is ψ-algebraic.

By [10, Proposition 2] (see Remark 2 of [10]) we have

Theorem 3.1. *With the above notation $\theta|_{G_x} \not\equiv 1$ for all $x \in X$.*

Proof of Theorem 1.1. The theorem follows from Theorems 2.5(2) and 3.1. □

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