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Prescribing the Webster scalar curvature on *CR* spheres

Préscription de la courbure scalaire de Webster sur la sphère CR

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A B S T R A C T

In this Note, we give two new perturbative results for prescribing the Webster scalar curvature on the $(2n+1)$-dimensional sphere endowed with its standard *CR* structure. The first result generalizes the one obtained by A. Malchiodi and F. Uguzzoni (2002) in [9].

R É S U M É

Dans cette Note, on donne deux nouveaux résultats perturbatifs pour préscrire la courbure scalaire de Webster sur la sphère de dimension $(2n+1)$ munie de sa structure *CR* standard. Le premier résultat généralise celui obtenu par A. Malchiodi et F. Uguzzoni (2002) dans [9].

1. Introduction

In this Note, we revisit a problem of geometric origin. Namely, let $\mathbb{S}^{2n+1}$ be the unit sphere of $\mathbb{C}^{n+1}$ defined by $\mathbb{S}^{2n+1} = \{\zeta = (\zeta^1, \dots, \zeta^{n+1}) \in \mathbb{C}^{n+1} / |\zeta|^2 = \sum_{j=1}^{n+1} |\zeta^j|^2 = 1\}$ endowed with its standard contact form $\theta = i(\bar\partial - \partial)|\zeta|^2 = i\sum_{j=1}^{n+1} \zeta^j \, d\bar\zeta^j - \bar\zeta^j \, d\zeta^j$. The Webster scalar curvature problem resumes to find a contact form $\tilde\theta$ *CR* equivalent to θ such that the associated Webster scalar curvature $R_{\tilde\theta} = K$, where K is some given function on $\mathbb{S}^{2n+1}$. It is equivalent to find a solution u of the following problem:

$$\begin{cases} Lu = Ku^{1+\frac{2}{n}}, \\ u > 0 \quad \text{on } \mathbb{S}^{2n+1} \end{cases} \tag{1}$$

where $L = \frac{2(n+1)}{n}\Delta_\theta + \frac{n(n+1)}{2}$ is the conformal sub-Laplacian (since the Webster scalar curvature of θ is $R_\theta = \frac{n(n+1)}{2}$), and Δ_θ – denoted also by $\Delta_\flat$ in the literature – is the sub-Laplacian operator (the real part of the Kohn Laplacian $\Box_\flat$) for $(\mathbb{S}^{2n+1}, \theta)$. Note that problem (1) is the *CR* counterpart of the scalar curvature problem in the Riemannian setting (see e.g. [2,8,10]).

This problem has been envisaged under perturbation or symmetric hypotheses (see [5] and [9]). Our aim is to handle such a question using some topological and dynamical tools related to the theory of critical points at infinity (see Bahri [1]).

Let $K : \mathbb{S}^{2n+1} \longrightarrow \mathbb{R}$ be a C^2 function. We introduce the following assumption:

(**N.D**) K has a finite set of nondegenerate critical points, denoted by $\mathcal{K}$, such that $\Delta_\theta K(y) \neq 0$, $\forall y \in \mathcal{K}$.

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We introduce the subset:

$$\mathcal{K}^+ = \{ y \in \mathcal{K};\ -\Delta_\theta K(y) > 0 \}, \tag{2}$$

and we denote by (**C**.**C**) the condition:

(**C**.**C**) Assume that $K(\zeta) = 1 + \varepsilon K_0(\zeta)$, $\forall \zeta \in \mathbb{S}^{2n+1}$, where $K_0 \in \mathcal{C}^2(\mathbb{S}^{2n+1})$ and $|\varepsilon|$ small.

Let $ind(K, y)$ be the Morse index of K at its critical point y. For an integer k, we write $k \in \aleph$, if k satisfies the following condition: For any $y \in \mathcal{K}^+$, we have $2n + 1 - ind(K, y) \neq k + 1$. That is

$$\aleph = \{ k \in \mathbb{N} / \forall y \in \mathcal{K}^+,\ 2n + 1 - ind(K, y) \neq k + 1 \}. \tag{3}$$

For example if $k \geqslant 2n + 1$ then $k \in \aleph$.

Our first existence result generalizes that of [9]:

Theorem 1.1. *Let $n \geqslant 1$, $K : \mathbb{S}^{2n+1} \to \mathbb{R}$ a $\mathcal{C}^2$ positive function satisfying* (**N**.**D**) *and* (**C**.**C**)*. If*

$$\max_{k \in \aleph} \Bigg| 1 - \sum_{\substack{y \in \mathcal{K}^+ \\ 2n+1-ind(K,y) \leqslant k}} (-1)^{2n+1-ind(K,y)} \Bigg| \neq 0 \tag{4}$$

then, for $|\varepsilon|$ small enough, there exists a solution for problem (1).

As a corollary we find the result of [9]:

Corollary 1.2. *Let $n \geqslant 1$. $K : \mathbb{S}^{2n+1} \to \mathbb{R}$ a $\mathcal{C}^2$ positive function satisfying* (**N**.**D**) *and* (**C**.**C**)*. If*

$$\sum_{y \in \mathcal{K}^+} (-1)^{2n+1-ind(K,y)} \neq 1, \tag{5}$$

then, for $|\varepsilon|$ small enough, there exists a solution for problem (1).

Our second existence result is not based on the count-indices formula. We introduce

$$\text{for } p \in \mathbb{N}, \quad \mathcal{K}_p^+ = \{ y \in \mathcal{K}^+;\ ind(K, y) = p \}, \quad \text{and} \quad \mathcal{K}_*^+ = \mathcal{K}^+ \setminus \mathcal{K}_0^+. \tag{6}$$

Theorem 1.3. *Let $n \geqslant 1$, $K : \mathbb{S}^{2n+1} \to \mathbb{R}$ a $\mathcal{C}^2$ positive function satisfying* (**N**.**D**)*,* (**C**.**C**)*, and* (**A**) *$\exists \tilde{y} \in \mathcal{K}_1^+$, such that, $K(\tilde{y}) \geqslant K(z)$, $\forall z \in \mathcal{K}_2^+$.*

Then, provided that $|\varepsilon|$ is small enough, there exists a solution for problem (1).

2. Variational framework. Lack of compactness. Proofs of the results

Let $\mathcal{S}^1$ be the completion of $\mathcal{C}^\infty(\mathbb{S}^{2n+1})$ for the norm $\|u\|^2 = \int_{\mathbb{S}^{2n+1}} Lu.u\, \theta \wedge d\theta^n$. Let $\sum = \{ u \in \mathcal{S}^1 / \|u\| = 1 \}$ and $\sum^+ = \{ u \in \sum / u \geqslant 0 \}$. The Euler functional associated to problem (1) on $\mathcal{S}^1$ is:

$$J(u) = \frac{\|u\|^2}{(\int_{\mathbb{S}^{2n+1}} K|u|^{2+\frac{2}{n}} \theta \wedge d\theta^n)^{\frac{n}{n+1}}}.$$

One knows that if v is a critical point of J in $\sum^+$, then $u = J(v)^{\frac{n}{2}} v$ is a solution for (1) in $\mathcal{S}^1$, and hence the contact form $\tilde{\theta} = u^{\frac{2}{n}} \theta$ has its Webster scalar curvature $R_{\tilde{\theta}} = K$.

Problem (1) is known to be delicate because the inclusion $\mathcal{S}^1 \hookrightarrow L^{\frac{2(n+1)}{n}}$ is continuous but not compact, and the functional J does not satisfy the Palais–Smale condition. In order to characterize the sequences failing the Palais–Smale condition, we recall some definitions and notations. Let ω be the solution of Yamabe problem on the Heisenberg group $\mathbb{H}^n$, defined for all $\xi = (z, t)$ in $\mathbb{H}^n$ by $\omega(\xi) = |1 + |z|^2 - it|^{-n}$. For each $(g, \lambda) \in \mathbb{H}^n \times (0, \infty)$ we obtain the other solutions $\omega_{(g,\lambda)}(\xi) = \lambda^n \omega(\lambda g^{-1} \xi)$. Now, for each $(a, \lambda) \in \mathbb{S}^{2n+1} \times (0, \infty)$, we introduce the solution of Yamabe problem on $\mathbb{S}^{2n+1}$:

$$\delta_{(a,\lambda)}(\zeta) = \frac{1}{|1 + \zeta^{n+1}|^n} \omega_{(F(a),\lambda)} \circ F(\zeta) \tag{7}$$

where F is a biholomorphic map from $\mathbb{S}^{2n+1} \setminus \{\text{a point}(\neq a)\}$ onto $\mathbb{H}^n$, induced by the Cayley Transform (see [7,9]). The sphere is globally CR equivalent to itself, then, standard arguments and results of [3] and [4] (where the case of manifolds locally CR-equivalent to the sphere was treated) are valid here. Since we are arguing by contradiction we have

Proposition 2.1. *Assume* (1) *has no solution. The only critical points at infinity of J in $\sum^+$ are combinations of q masses ($q \geqslant 1$), $\sum_{i=1}^{q} \delta_{(y_i,+\infty)} := (y_1, \ldots, y_q)_\infty$, where the y_i's are distinct in $\mathcal{K}^+$.*

The Morse index of $(y_1, \ldots, y_q)_\infty$ is $m(y_1, \ldots, y_q)_\infty = q - 1 + \sum_{i=1}^{q} 2n + 1 - ind(K, y_i)$.

Proof of Theorem 1.1. Since $K = 1 + \varepsilon K_0$, for $\varepsilon = 0$ we obtain the Yamabe functional J_0, which possesses a $(2n+2)$-dimensional manifold of critical points $\mathcal{Z} = \{\delta_{(a,\lambda)},\ (a,\lambda) \in \mathbb{S}^{2n+1} \times (0,\infty)\}$. For $\beta \in \mathbb{R}$, let $J^\beta = \{u \in \sum^+,\ J(u) \leqslant \beta\}$. Observe that, since K_0 is bounded, $J(u) = J_0(u)(1 + O(\varepsilon))$, where $O(\varepsilon)$ is independent of u and tends to 0 with ε. Readily we derive:

Lemma 2.1. *Let $\eta > 0$, for $|\varepsilon|$ small enough, we have $J^{S+\eta} \subset J_0^{S+2\eta} \subset J^{S+3\eta}$.*

Here S is the best Sobolev constant $S = J_0(\delta_{(a,\lambda)}) = \min J_0$ on $\sum^+$. On the other hand, the critical level $J((y_1, \ldots, y_q)_\infty) = S(\sum_{i=1}^{q} 1/K(y_i)^n)^{1/(n+1)}$ tends to $Sq^{1/(n+1)}$ as $\varepsilon \to 0$, since $K(y_i) = 1 + \varepsilon K_0(y_i)$. Taking $\eta = S/4$, we can assume $|\varepsilon|$ sufficiently small that: critical points at infinity made of two bubbles or more are above the level $S + 3\eta$, and those made of a single bubble are below the level $S + \eta$. Therefore, J has no critical points at infinity in the set $J_{S+\eta}^{S+3\eta} = \{u \in \sum^+,\ S + \eta \leqslant J(u) \leqslant S + 3\eta\}$. Since, arguing by contradiction, we assume that (1) has no solution, it follows that $J^{S+3\eta} \simeq J^{S+\eta}$, where $\simeq$ denotes retracts by deformation. Using Lemma 2.1, we have that $J_0^{S+2\eta} \simeq J^{S+\eta}$. Now, we claim that $J^{S+\eta}$ is a contractible set. Indeed, from what precedes, it is sufficient to prove that $J_0^{S+2\eta}$ is a contractible set. Let $u_0 \in J_0^{S+2\eta}$, and $s \mapsto u(s, u_0)$ the Yamabe flow line. The flow verifies the following equation:

$$\begin{cases} \dfrac{\partial u}{\partial s} = -\partial J_0(u), \\ u(0) = u_0. \end{cases}$$

Using the results of [6], we know that the Palais–Smale condition is satisfied for the above equation for all $s > 0$. When $s \to +\infty$, $u(s, u_0)$ converges to a single mass in $\mathcal{Z}$. Thus, $J_0^{S+2\eta} \simeq \mathcal{Z}$. It follows that $J_0^{S+2\eta}$ is a contractible set, since $\mathcal{Z}$ is a contractible set. Our claim follows. Now, let ℓ be the integer for which $\max_{k \in \aleph} |1 - \sum_{\{y \in \mathcal{K}^+,\, m(y)_\infty \leqslant k\}} (-1)^{m(y)_\infty}| \neq 0$ is achieved. Here $m(y)_\infty = 2n + 1 - ind(K, y)$ is the Morse index of the critical point at infinity $(y)_\infty$. Let $X_\ell^\infty = \bigcup_{\{y \in \mathcal{K}^+,\, m(y)_\infty \leqslant \ell\}} \overline{W_u(y)_\infty}$. It is a stratified set of top dimension ℓ, and, since it is made of unstable manifolds of critical points at infinity of a single mass, we derive from what precedes that $X_\ell^\infty \subset J^{S+\eta}$. Observe that X_ℓ^∞ is contractible in $J^{S+\eta}$, since $J^{S+\eta}$ is a contractible set. More precisely, there exists a contraction $h : [0,1] \times X_\ell^\infty \to J^{S+\eta}$, i.e. h continuous and such that $h(0, u) = u$ and $h(1, u) = \tilde{u}$ a fixed point in X_ℓ^∞. Let $H = h([0,1] \times X_\ell^\infty)$. H is a contractible stratified set of dimension $\ell + 1$. Using the flow lines of $-\partial J$, and the fact that $H \subset J^{S+\eta}$, we have $H \simeq \bigcup_{\{y \in \mathcal{K}^+,\, m(y)_\infty \leqslant \ell+1\}} W_u(y)_\infty$. Now, using the fact that $\ell \in \aleph$, there are no critical points at infinity of Morse index $\ell + 1$. We derive that $H \simeq X_\ell^\infty$. Then, taking the Euler characteristic of both sides, we derive that $1 = \sum_{\{y \in \mathcal{K}^+,\, m(y)_\infty \leqslant \ell\}} (-1)^{m(y)_\infty}$. This contradicts the assumption of Theorem 1.1. □

Proof of Theorem 1.3. Arguing by contradiction, we assume that (1) has no solution. Let ∂ be the boundary operator in the sense of Floer–Milnor homology as introduced in [11]. We recall that singular chains, in this homology, are generated by unstable manifolds of critical points of J, and, if $(y)_\infty$ is a critical point at infinity of Morse index $m(y)_\infty$, then

$$\partial\big(W_u(y)_\infty\big) = \sum_{\{(z)_\infty;\ m(z)_\infty = m(y)_\infty - 1\}} i\big((y)_\infty, (z)_\infty\big) W_u(z)_\infty$$

where $i((y)_\infty, (z)_\infty)$ is the intersection number of $W_u(y)_\infty$ and $W_s(z)_\infty$, the unstable (resp. stable) manifold of $(y)_\infty$ (resp. $(z)_\infty$), with respect of $-\partial J$ (see [11]).

Taking $y = \tilde{y}$, defined in assumption (**A**) of Theorem 1.3, $W_u(\tilde{y})_\infty$ is a manifold of dimension $m(\tilde{y})_\infty = 2n$, and satisfies $W_u(\tilde{y})_\infty \cap W_s(z)_\infty = \emptyset$ for any $(z)_\infty$ of Morse index $2n - 1$, since under assumption (**A**), $J((\tilde{y})_\infty) \leqslant J((z)_\infty)$ for all $z \in \mathcal{K}_2^+$. It follows that $\partial(W_u(\tilde{y})_\infty) = 0$, and hence $W_u(\tilde{y})_\infty$ defines a cycle in $C_{2n}(X^\infty)$ the group of $2n$-dimensional chains of $X^\infty = \bigcup_{y \in \mathcal{K}_*^+} W_u(y)_\infty$. Note that X^∞ is a stratified set of top dimension $2n$, since the highest Morse index of critical points at infinity $(y)_\infty$ where $y \in \mathcal{K}_*^+$, is less than or equal to $2n$. But, $W_u(\tilde{y})_\infty$ cannot be in the boundary of a $2n$-dimensional chain of X^∞. Therefore $W_u(\tilde{y})_\infty$ defines a homological class of dimension $2n$ which is nontrivial in X^∞. Denoting by $H_{2n}(X^\infty)$ the $2n$-dimensional homology group of X^∞, we then have

$$H_{2n}\big(X^\infty\big) \neq 0. \tag{8}$$

Using the same arguments and notations of the proof of Theorem 1.1, we derive that X^∞ is contractible in $J^{S+\eta}$ which retracts by deformation on X^∞. It follows that X^∞ is a contractible set, and therefore $H_k(X^\infty) = 0,\ \forall k \geqslant 1$, which is in contradiction with (8). This ends the proof of Theorem 1.3. □

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