C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1191–1194

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Differential Geometry

On compact Finsler spaces of positive constant curvature

Sur les espaces finsleriennes compacts à courbure positive constante

Behroz Bidabad

Department of Mathematics, Amirkabir University of Technology (Tehran Polytechnic), Tehran 15914, Iran

ARTICLE INFO

Article history:
Received 5 September 2011
Accepted 18 October 2011
Available online 4 November 2011

Presented by the Editorial Board

ABSTRACT

An n-dimensional ($n \geqslant 2$) simply connected, compact without boundary Finsler space of positive constant sectional curvature is conformally homeomorphic to an n-sphere in the Euclidean space $\mathbb{R}^{n+1}$.

RÉSUMÉ

Un espace de Finsler de dimension n ($n \geqslant 2$), simplement connexe, compact, non-borné et à courbure sectionnelle positive constante est conformément homéomorphe à une n-sphère d'un espace euclidien $\mathbb{R}^{n+1}$.

1. Introduction

The normal coordinates have been proved to be an extremely useful tool in the theory of global Riemannian geometry, while it is not so useful in Finsler geometry. In fact, in the latter case the exponential map is only C^1 at the zero section of TM while in the former case it is C^∞. More intuitively H. Akbar-Zadeh proved that exponential map is C^2 at the zero section if and only if the Finsler structure is of Berwald type, cf. [1]. In 1950s Y. Tashiro has defined adapted coordinates which were used by many authors to obtain global results in Riemannian geometry, which will not be mentioned here. In fact, the definition of this coordinate system on a manifold M is equivalent to the existence of a non-trivial solution for a certain second order differential equation which describes circles preserving transformations on M. Recently, the circle-preserving transformations have been studied in Finsler geometry by present author and Z. Shen, cf. [5]. Previously, inspired by the work of Tashiro, cf. [8], the present author has specialized adapted coordinates to Finsler setting in a joint paper, and proved the following theorem, cf. [2].

Theorem A. *Let (M, g) be a connected complete Finsler manifold of dimension $n \geqslant 2$. If M admits a non-trivial solution of*

$$\nabla_i \nabla_j \rho = \phi g_{ij}, \tag{1}$$

where, ∇ is the Cartan h-covariant derivative, then depending on the number of critical points of ρ, i.e. zero, one or two respectively, it is conformal to (a) *A direct product $J \times \overline{M}$ of an open interval J of the real line and an $(n-1)$-dimensional complete Finsler manifold $\overline{M}$.* (b) *An n-dimensional Euclidean space.* (c) *An n-dimensional unit sphere in a Euclidean space.*

Here, we show that if (M, g) is compact then only the third case may occur. More precisely we prove:

E-mail address: bidabad@aut.ac.ir.

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doi:10.1016/j.crma.2011.10.014

Theorem 1. *Let (M, g) be an n-dimensional ($n \geqslant 2$) Finsler manifold, compact simply connected and without boundary. In order that (M, g) admits a non-trivial solution ρ of Eq.* (1)*, it is necessary and sufficient that (M, g) be conformally homeomorphic to a standard n-sphere in the Euclidean space $\mathbb{R}^{n+1}$.*

Theorem 2. *Let (M, g) be an n-dimensional ($n \geqslant 2$) Finsler manifold, compact simply connected and without boundary, of positive constant flag curvature. Then it is conformally homeomorphic to a standard n-sphere in the Euclidean space $\mathbb{R}^{n+1}$.*

2. Preliminaries

Let M be a real n-dimensional manifold of class C^∞. We denote by $TM \to M$ the bundle of tangent vectors and by $\pi : TM_0 \to M$ the fiber bundle of non-zero tangent vectors. A *Finsler structure* on M is a function $F : TM \to [0, \infty)$, with the following properties: F is differentiable (C^∞) on TM_0; F is positively homogeneous of degree one in y, i.e. $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$, where we denote an element of TM by (x, y). The Hessian matrix of F^2 is positive definite on TM_0, i.e. $(g_{ij}) := (\frac{1}{2}[\frac{\partial^2}{\partial y^i \partial y^j} F^2])$. A *Finsler manifold* (M, g) is a pair of a differentiable manifold M and a tensor field $g = (g_{ij})$. Let (x, y) be the line element of TM and $P(y, X) \subset T_x(M)$ a 2-plane generated by the vectors y and X in T_xM. Then the *sectional* or *flag* curvature $K(x, y, X)$ with respect to the plane $P(y, X)$ at a point $x \in M$ is defined by $K(x, y, X) := \frac{g(R(X,y)y,X)}{g(X,X)g(y,y) - g(X,y)^2}$, where $R(X, y)y$ is the h-curvature tensor of Cartan connection. If K is independent of X, then (M, g) is called *space of scalar curvature*. If K has no dependence on x or y, then the Finsler manifold is said to be of *constant curvature*. We say that a curve γ on M is a *geodesic* of a Finsler connection ∇, if its natural lift $\tilde{\gamma}$ to TM, is a geodesic of ∇, or equivalently $\nabla_{\frac{d\tilde{\gamma}}{dt}} \frac{d\tilde{\gamma}}{dt} = 0$. Two points p and q are said to be *conjugate points* along a geodesic γ if there exists a non-zero Jacobi field along γ that vanishes at p and q, cf. [3]. Let $\rho : M \to [0, \infty)$ be a scalar function on M and $\nabla_i \nabla_j \rho = \phi g_{ij}$, a second order differential equation, where ∇_i is the Cartan horizontal covariant derivative and ϕ is a function of x alone, then we say that Eq. (1) has a solution ρ. The solution ρ is said to be *trivial* if it is constant. Existence of solution of Eq. (1) is equivalent to the existence of some special conformal change of metric on M. We denote by $\mathtt{grad}\,\rho = \rho^i \partial/\partial x^i$ the gradient vector field of ρ, where $\rho^i = g^{ij}\rho_j$, $\rho_j = \partial\rho/\partial x^j$ and $i, j, \ldots$ run over the range $1, \ldots, n$. We say the point o of (M, g) is a *critical point* of ρ if the vector field $\mathtt{grad}\,\rho$ vanishes at o, or equivalently if $\rho'(o) = 0$, where $\rho' = d\rho/dt$. All other points are called *ordinary points* of ρ on M. It's noteworthy to recall that the partial derivatives ρ_j are defined on the manifold M, while ρ^i the components of $\mathtt{grad}\,\rho$ are defined on the slit tangent bundle TM_0. Hence, $\mathtt{grad}\,\rho$ can be considered as a section of $\pi^*TM \to TM_0$, the pulled-back tangent bundle over TM_0, and its trajectories lie on TM_0. Let the Finsler manifold (M, g) admits a non-trivial solution ρ of (1), then for any ordinary point $p \in M$ there exists a coordinate neighborhood $\mathcal{U}$ of p which contains no critical point, and where we can choose a system of coordinates $(u^1 = t, u^2, \ldots, u^n)$ having the following properties, cf. [2];

- The function ρ depends only on the first variable $u^1 = t$ on $\mathcal{U}$.
- The integral curve of $\mathtt{grad}\,\rho$ is a geodesic and geodesic containing such a curve is called a *ρ-curve* or a *t-geodesic* of ρ.
- The connected component of a regular hyper-surface defined by $\rho = \text{constant}$, is called a *level set of ρ* or simply a t-level. Given a solution ρ and a point $q \in \mathcal{U}$, there exists one and only one t-level set of ρ passing through q. The t-geodesics form the normal congruence to the family of t-level sets of ρ.
- The curves defined by $u^\alpha = \text{const}$ are t-geodesics of ρ, and the parameter $u^1 = t$ may be regarded as the arc-length parameter of t-geodesics.
- The components g_{ij} of the Finsler metric tensor g satisfy $g_{\alpha 1} = g_{1\alpha} = 0$, where the Greek indices α, run over the range $2, 3, \ldots, n$ and the Latin indices i, j, run over the range $1, 2, \ldots, n$.
- In adapted coordinates the first fundamental form of (M, g) is given by

  $$ds^2 = (dt)^2 + \rho'^2 f_{\gamma\beta}\, du^\gamma\, du^\beta, \tag{2}$$

  where $f_{\gamma\beta}$, given by $g_{\gamma\beta} = \rho'^2 f_{\gamma\beta}$, are components of a metric tensor on a t-level of ρ and $g_{\gamma\beta}$ is the induced metric tensor of this t-level. For more details about our purpose on adapted coordinates, one can refer to [2,4,8].

Throughout this paper, all manifolds are supposed to be connected.

3. Compact Finsler spaces of constant curvature

Proof of Theorem 1. Let (M, g) be a an n-dimensional, $n \geqslant 2$, Finsler manifold which admits a non-trivial C^∞ solution ρ of Eq. (1). Consider the so-called t-geodesic which is an integral curve of the gradient vector field $\mathtt{grad}\,\rho$ on M. It is well known that every t-geodesic is a geodesic on M. Since M is compact by extension of Extreme Value Theorem to differentiable manifolds every solution ρ of Eq. (1) is bounded and attains its extremum values on M. Once the assumption is made as M is without boundary, differentiability of ρ requires that these extremum values are critical points. Let O be a critical point for a t-geodesic on M. By compactness, M must have finite diameter D and no t-geodesic longer than D

can remain minimizing. Thus every t-geodesic longer than D issuing from O contains at least two critical points. Before proceeding further, we shall recall that on a Finsler manifold there is no more than two critical points of ρ on every t-geodesic emanating from O, cf. [2]. Therefore, every t-geodesic on (M,g) contains exactly two critical points. Thus by means of Theorem A, (M,g) is conformal to an n-dimensional sphere in the Euclidean space $\mathbb{R}^{n+1}$, with the first fundamental form (2). Moreover, M is assumed to be simply connected and an extension of the Milnor theorem to Finslerian category, cf. [6], implies that M is topologically homeomorphic to an n-sphere.

Conversely, let (M,g) be compact and conformally homeomorphic to the n-sphere $S^n \subset \mathbb{R}^{n+1}$. The first fundamental form of S^n is given by

$$g_{S^n} = \mathrm{d}t^2 + \sin^2 t g_{S^{n-1}}, \tag{3}$$

where $g_{S^{n-1}}$ is the first fundamental form of the hypersphere S^{n-1}, cf. [7]. Let $\gamma := x^i(t)$ be a geodesic on (M,g), by definition its differential equation is given by

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = \varphi\frac{\mathrm{d}x^i}{\mathrm{d}t}, \tag{4}$$

where t is an arbitrary parameter and φ is a function of t. If we denote $\frac{\mathrm{d}x^j}{\mathrm{d}t} = \gamma^j$, by virtue of Eq. (4) we have

$$\gamma^k\frac{\mathrm{d}\gamma^l}{\mathrm{d}x^k} + \Gamma^l_{jk}\gamma^j\gamma^k = \varphi\gamma^l. \tag{5}$$

This is equivalent to $\gamma^k(\nabla_k\gamma^l) = \varphi\gamma^l$, where ∇_k is the Cartan h-covariant derivative. Denoting $\gamma_i := g_{il}\gamma^l$ and contracting with g_{il}, we obtain $\gamma^k(\nabla_k\gamma_i) = \varphi\gamma_i$, which leads to

$$\gamma^k(\nabla_k\gamma_i - \varphi g_{ik}) = 0. \tag{6}$$

The assumption that (M,g) is conformally homeomorphic to the standard sphere (S^n, g_{S^n}) implies that the Finsler metric g is positively proportional to g_{s_n}, that is $g = e^{2\psi}g_{S^n}$ where, by the Knebelman theorem ψ is a function on M. Therefore g is also a function on M and hence a Riemannian metric. By compactness of M the vector field γ^k is complete and Eq. (6) leads to $\nabla_k\gamma_i = \varphi g_{ik}$ which is equivalent to Eq. (1). This completes the proof of theorem. □

Proposition 3.1. *Let (M,g) be an n-dimensional compact Finsler manifold of constant flag curvature K. Then the following SODE*

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}t} + K\rho = 0, \tag{7}$$

admits a non-trivial solution on (M,g) if and only if $K > 0$.

Proof. Let (M,g) be a Finsler manifold of constant flag curvature K, then the following equation holds well, cf. [1], see also [3]:

$$\ddot{A}_{ijk} + KA_{ijk} = 0, \tag{8}$$

where A_{ijk} is the Cartan torsion tensor, $\dot{A}_{ijk} := (\nabla_s A_{ijk})\ell^s$ and $\ddot{A}_{ijk} := (\nabla_s\nabla_t A_{ijk})\ell^s\ell^t$ and $\ell^i := y^i/F$. Let $\gamma : \mathbb{R} \to M$ be any geodesic parameterized by arc length t on (M,g) passing through $\gamma(0) = p$, having tangent vector $\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \ell^i$ and the canonical lift $\hat{\gamma} := \frac{\mathrm{d}\gamma}{\mathrm{d}t}$ to TM_0. To differentiate the Cartan torsion tensor along γ, we consider the linearly independent parallel sections $X(t)$, $Y(t)$ and $Z(t)$ of π^*TM along $\hat{\gamma}$ with $X(0) = X$, $Y(0) = Y$ and $Z(0) = Z$ at the point p. By a direct computation, for two linearly independent parallel vector fields $X(t)$ and $Y(t)$ along γ, we have $\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{\alpha}(t)}(X(t), Y(t)) = 0$, see for instance [1] or [3]. In this sense, a g-orthonormal basis for π^*TM remains g-orthonormal at every point $(x(t), y(t))$ along $\hat{\gamma}$. Therefore, by assuming $A(t) = A(X(t), Y(t), Z(t))$, $\dot{A}(t) = \dot{A}(X(t), Y(t), Z(t))$ and $\ddot{A}(t) = \ddot{A}(X(t), Y(t), Z(t))$ along γ, we have $\frac{\mathrm{d}A}{\mathrm{d}t} = \dot{A}$, $\frac{\mathrm{d}\dot{A}}{\mathrm{d}t} = \ddot{A}$ and Eq. (8) becomes

$$\frac{\mathrm{d}^2A(t)}{\mathrm{d}t^2} + KA(t) = 0. \tag{9}$$

By assuming $\rho(t) = A(t)$ we obtain Eq. (7). To complete the proof we consider three cases.

Case. $K > 0$. In this case $\rho(t) = a\cos\sqrt{K}t + b\sin\sqrt{K}t$ is a non-trivial general solution for Eq. (7).

Case. $K < 0$. In this case the general solution is given by

$$A(t) = \alpha e^{\sqrt{-K}t} + \beta e^{-\sqrt{-K}t}. \tag{10}$$

For $v \in TM_0$, assume that the norm of Cartan torsion tensor is $\|A\|_v := \sup A(X, Y, Z)$, where the supremum is taken over all unit vectors of π_v^*TM. Suppose that $S_xM = \{w \in T_xM,\ F(w) = 1\}$ is the indicatrix and $\|A\| = \sup_{v\in SM} \|A\|_v$, where $SM = \bigcup_{x\in M} S_xM$. Since M is compact the norm $\|A\|$ is bounded. On the other hand M is compact and therefore geodesically complete and the parameter t takes all the values in $(-\infty, +\infty)$. Letting $t \to +\infty$ or $t \to -\infty$, then Eq. (10) implies that $A(0) = 0$. In fact as t approaches to $t \to \pm\infty$ the left hand side of the equation is bounded and the right hand side is infinity, so Eq. (10) can hold only if the coefficients α and β vanish. Replacing it in Eq. (10), we obtain $A(t) = 0$. That is to say the solution $\rho(t) = A(t)$ of Eq. (7) is trivial.

Case. $K = 0$. In this case the general solution of Eq. (7) is given by $A(t) = \alpha + \beta t$, where α and β are constants. Following the procedure described above we obtain $\beta = 0$ which implies that $A(t) = \alpha$, is constant and hence a trivial solution of Eq. (7). This completes the proof of the proposition. □

Proof of Theorem 2. Let (M, g) be a compact Finsler manifold of positive constant flag curvature K. If we assume $\phi = -K\rho$ then Eq. (1) reduces to

$$\nabla_i\nabla_j\rho + K\rho g_{ij} = 0. \tag{11}$$

Following an argument similar to that in the proof of the above proposition, Eq. (11) reduces to Eq. (7) along geodesics. By virtue of Proposition 3.1 there is a non-trivial solution say ρ for Eq. (7) and hence for Eq. (11) on M. Therefore (M, g) admits a non-trivial solution ρ for Eq. (1). A simple application of Theorem 1 completes the proof of Theorem 2. □

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