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Partial Differential Equations

Uniqueness for an ill-posed parabolic system

Unicité pour un système parabolique mal-posé

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ABSTRACT

The purpose is the uniqueness for an ill-posed parabolic system. This result enables us to state the identifiability for the problem of detecting pointwise organic pollution sources in surface waters.

RÉSUMÉ

L'objectif est de prouver l'unicité de solution pour un système parabolique mal-posé. Ce résultat sert à établir l'identifiabilité pour le problème de detection de sources ponctuelles de pollution organique dans les eaux de surface.

Version française abrégée

Deux traceurs sont couramment employés dans l'analyse et la gestion des eaux. La Demande Biochimique en Oxygène (DBO) est largement utilisée comme mesure du degré de pollution, causée par des agents organiques, duquel dépendent les caractéristiques organoleptiques de l'eau (odeur, couleur, saveur). Un autre indicateur important à considérer est l'Oxygène Dissous (OD) nécessaire à la vie aquatique et aux bactéries aérobies, acteurs de la biodégradation de la matière organique. L'objectif est de reconstruire les sources de pollution organique qui alimentent l'équation sur la DBO à partir d'observations sur le déficit en OD. L'avantage de cette méthodologie réside dans le fait que les mesures sur l'OD sont instantannées alors que celles sur la DBO doivent suivre un protocole chimique strict qui peut durer jusqu'à cinq jours. Ce protocole ressemble à celui d'un sac qu'on vide pour en connaître le contenu.

Des modèles bien établis pour les rivières conduisent à un système aux limites de réaction-dispersion monodimensionnel. Celui qui nous préoccupe est fourni par (7). Les caractéristiques des sources apparaissant dans les deux premières équations, à savoir la position supposée fixe des sources (r, s) et leurs intensités $(f(\cdot), g(\cdot))$ variables en temps, sont des inconnues du problème. Pour compenser ce manque d'information sur $F = (r, f(\cdot))$ et $G = (s, g(\cdot))$, les ingénieurs hydrologues effectuent des observations sur l'OD au niveau de deux stations encadrant les sources F et G. L'identifiabilité pour ce problème de détection de sources consiste à établir qu'un jeu d'observations (8) détermine au plus un couple de sources (F, G). L'application immédiate de ce résultat est qu'il permet de discriminer l'agent polluant.

La méthodologie pour obtenir un tel résultat d'identifiabilité consiste d'abord à reconstruire la densité de DBO au niveau des stations d'observation. Cela passe par l'analyse du système parabolique (1), avec deux conditions à la limite imposées sur la densité d'OD à l'emplacement des observations et aucune sur la concentration de DBO. Un résultat d'unicité est alors démontré grâce à des outils variationnels de la théorie de point-selle développée dans [3,2], et particulièrement à l'aide

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d'un théorème d'unicité dû à A. Pazy [5] en théorie des semi-groupes. C'est l'objet du Théorème 2.4. Cette unicité renforcée par un raisonnement identique à celui de [1] permet d'obtenir l'identifiabilité espérée s'agissant de la détection des sources dans le modèle (7) à l'aide des observations (8). Ce résultat est énoncé dans le Théorème 3.2.

1. Introduction

We are concerned with a uniqueness result for a coupled time-dependent dispersion–reaction system. This problem arises from the Biochemical Oxygen Demand–Dissolved Oxygen model[1] governing the interaction between organic pollutants and the oxygen available in stream waters. The absence of any prescribed condition on the BOD density is compensated by observations on the DO concentration which provide over-determined Cauchy boundary conditions. We pursue the reconstruction of the BOD flux at the boundary. That problem turns out to be ill-posed. However, we are able to state a uniqueness result owing to a suitable saddle-point variational framework (see [3,2]) and to a uniqueness theorem proved by A. Pazy (see [5]). An important application of such a result resides in the identifiability of point-wise organic pollution sources in rivers from observations only on the DO concentration.

2. An ill-posed dispersion–reaction system

In the water quality modeling, two tracers are currently used to quantify the organic pollution in a body of water, the BOD denoted by $b(\cdot,\cdot)$ and the DO which pointed at by $c(\cdot,\cdot)$. Let I be the segment $(0,\xi)$ and $T>0$ is a fixed real number. The system on the concentrations (b,c) reads as follows

$$\begin{aligned}
&\partial_t b - \big(d_* b'\big)' + r_* b = 0 \quad \text{in } I\times(0,T),\\
&\partial_t c - \big(dc'\big)' + rc - r_* b = 0 \quad \text{in } I\times(0,T),\\
&b(x,0) = c(x,0) = (0,0) \quad \text{in } I,\\
&b(0,t) = c(0,t) = (0,0) \quad \text{in } (0,T),\\
&\big(c(\xi,t), dc'(\xi,t)\big) = \big(\alpha(t),\beta(t)\big) \quad \text{in } (0,T).
\end{aligned} \tag{1}$$

Here, $c(\cdot,\cdot)$ is rather the deficit of DO which is the gap to the saturation level. (α,β) are given functions in time. The dispersion coefficients (d,d_*) and the reaction parameters (r,r_*) belong to $L^\infty(I)$, are positive and bounded away from zero. The particularity here is that no boundary conditions are provided on $b(\cdot,\cdot)$ at point $x=\xi$ while $c(\cdot,\cdot)$ enjoys both Dirichlet and Neumann conditions. This fact induces a strong coupling between the concentrations $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ and implies the necessity of a mixed formulation for the study, as will be seen below. This system turns out to be ill-posed and only the uniqueness can be handled successfully.

2.1. An abstract formulation

To investigate the uniqueness, we follow the approach of the semi-groups theory and we need to put the coupled system (1) into an abstract form. Let f and g be given in $L^2(I\times(0,T))$, consider the time-dependent system

$$\partial_t(b,c) + A(b,c) = (f,g), \tag{2}$$

where A is an unbounded linear operator defined in $\boldsymbol{L}^2(I) = L^2(I)^2$. The domain $D(A)\subset \boldsymbol{L}^2(I)$ is given by

$$D(A) = \big\{(\varphi,\psi)\in \boldsymbol{H}^1(I),\ \big(\big(d_*\varphi'\big)', \big(d\psi'\big)'\big)\in \boldsymbol{L}^2(I),\ (\varphi,\psi)(0) = (0,0),\ \big(\psi, d\psi'\big)(\xi) = (0,0)\big\},$$

and the operator A is defined to be

$$A(b,c) = \big(-\big(d_* b'\big)' + r_* b, -\big(dc'\big)' + rc - r_* b\big).$$

Notice that A is densely defined. Uniqueness for the time-dependent equation (2) is tightly connected to the resolvent $R(\lambda) = (\lambda + A)^{-1}$, for $\lambda > 0$ (see [4, Chapter XVII]). We are therefore primarily concerned with the existence and the boundedness of $R(\lambda)$ in $\boldsymbol{L}^2(I)$. To prove such a statement, we begin by the steady equation. Let us consider the problem

$$A(b,c) = (f,g), \tag{3}$$

for a given $(f,g)\in \boldsymbol{L}^2(I)$. The well-posedness may be achieved after laying down a suitable mixed variational framework (see [3, Chapter II]).

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[1] The acronym BOD-DO model is currently used.

2.2. Saddle-point variational formulation

Eq. (3) looks like the Stokes problem expressed in the stream function-vorticity variables. It may be put under a mixed variational formulation. Let us then set $Q = H_0^1(I)$ and define V as follows:

$$V = \{\varphi \in H^1(I),\ \varphi(0) = 0\}.$$

We introduce also the three continuous bilinear forms

$$a(b,\varphi) = -\int_I r_* b\varphi\,\mathrm{d}x, \quad \forall (b,\varphi) \in V \times V,$$

$$m_{(*)}(\varphi,\psi) = \int_I d_{(*)}\psi'\varphi'\,\mathrm{d}x + \int_I r_{(*)}\psi\varphi\,\mathrm{d}x, \quad \forall(\varphi,\psi) \in V \times Q.$$

The index $_{(*)}$ means that we need to use two mixed forms $m_*(\cdot,\cdot)$ and $m(\cdot,\cdot)$. They are respectively obtained by putting $_*$ for $m_*(\cdot,\cdot)$ and canceling it for $m(\cdot,\cdot)$. With these definitions, we can write (3) under a variational form that reads as follows: *find* $(b,c) \in V \times Q$ *verifying*

$$m_*(b,\psi) = (f,\psi)_{L^2}, \quad \forall \psi \in Q, \tag{4}$$

$$m(\varphi,c) + a(b,\varphi) = (g,\varphi)_{L^2}, \quad \forall \varphi \in V. \tag{5}$$

It is a non-symmetric mixed problem that fits into the saddle-point theory developed in [2]. The point now is to consider the existence, uniqueness and stability for problem (4)–(5). We need hence to state that each of the mixed forms $m(\cdot,\cdot)$ and $m_*(\cdot,\cdot)$ satisfies an inf-sup condition in $Q \times V$. Then, we turn to the bilinear form $a(\cdot,\cdot)$ which has to enjoy two inf-sup conditions on the null-spaces $\mathcal{N}$ and $\mathcal{N}_*$ defined by

$$\mathcal{N}_{(*)} = \{\varphi \in V,\ m_{(*)}(\varphi,\psi) = 0,\ \forall \psi \in Q\}.$$

These requirements are fulfilled and we have that

Lemma 2.1. *The bilinear form* $a(\cdot,\cdot)$ *satisfies the inf-sup conditions*

$$\inf_{\psi\in\mathcal{N}_*}\sup_{\varphi\in\mathcal{N}} \frac{a(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geqslant \eta_*, \qquad \inf_{\psi\in\mathcal{N}}\sup_{\varphi\in\mathcal{N}_*} \frac{a(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geqslant \eta.$$

The bilinear form $m_{(*)}(\cdot,\cdot)$ *satisfies the inf-sup condition in* $V \times Q$,

$$\inf_{\psi\in Q}\sup_{\varphi\in V} \frac{m_{(*)}(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geqslant \beta_{(*)}.$$

The statements on $m(\cdot,\cdot)$ and $m_*(\cdot,\cdot)$ are easy to obtain while the ones on $a(\cdot,\cdot)$ are obtained after remarking that $\dim\mathcal{N}_* = \dim\mathcal{N} = 1$ and $a(\varphi,\psi) = 0$, with $(\varphi,\psi) \in \mathcal{N}_* \times \mathcal{N}$, yields by the maximum principle that $\varphi = \psi = 0$. Necessary tools are thus all available for the well-posedness of the problem.

Lemma 2.2. *The mixed problem* (4)–(5) *has a unique solution* $(b,c) \in V \times Q$ *such that*

$$\|b\|_{H^1} + \|c\|_{H^1} \leqslant C\big(\|f\|_{L^2} + \|g\|_{L^2}\big).$$

The next important point to discuss is the existence of the resolvent $R(\lambda)$, for $\lambda > 0$. Let (f,g) be given in $\boldsymbol{L}^2(I)$, we investigate the equation

$$(\lambda + A)(b_\lambda, c_\lambda) = (f,g). \tag{6}$$

Similar to the steady equation (3), this system may be put under a mixed variational form. Handling the corresponding saddle-point problem yields the following result. Eq. (6) has a unique solution $(b_\lambda, c_\lambda) \in V \times Q$ and

$$\|b_\lambda\|_{L^2} + \lambda\|c_\lambda\|_{L^2} \leqslant C\big(\|f\|_{L^2} + \|g\|_{L^2}\big).$$

Above all, we have the following important stability result on the resolvent $R(\lambda)$:

Proposition 2.3. *There holds that*

$$\big\|R(\lambda)\big\|_{(\boldsymbol{L}^2(I)\to\boldsymbol{L}^2(I))} \leqslant C'.$$

The constant C' *is independent of* λ.

2.3. A uniqueness result

The bound provided in Proposition 2.3 suggests that the norm of the resolvent $R(\lambda)$ may not decay like λ^{-1}, for large λ. Actually it does not. Constructing a counter-example is possible. This confirms, if still needed, the failure of the Hille–Yosida Theory. Nevertheless, when confronted only with the uniqueness, a theorem stated by A. Pazy (see [5, Chapter 4, Theorem 1.2]) turns out to be sufficient. Pazy's theorem tells that provided that

$$\limsup_{\lambda\to+\infty} \frac{1}{\lambda} \log\left\| R(\lambda)\right\|_{(\boldsymbol{L}^2(I)\to \boldsymbol{L}^2(I))} = 0,$$

the initial value problem (2) has at most one solution. Seen that the resolvent $R(\lambda)$ satisfies that sufficient condition by Proposition 2.3, we have the following key result.

Theorem 2.4. *Problem* (2) *has at most one solution.*

3. Application to the identifiability of point-wise sources

The previous result may be used to state the identifiability for the recovery of the pollution point-wise sources from observations on the dissolved oxygen. The inverse problem to deal with is based on the Streeter–Phelps model completed by the dispersion terms (see [6])

$$\begin{aligned}
&\partial_t b - \left(d_* b'\right)' + r_* b = f(t)\delta(x-r) \quad \text{in } (0,L)\times(0,T),\\
&\partial_t c - \left(d c'\right)' + r c - r_* b = g(t)\delta(x-s) \quad \text{in } (0,L)\times(0,T),\\
&b(0,t) = c(0,t) = 0 \quad \text{in } (0,T),\\
&b(x,0) = c(x,0) = 0 \quad \text{in } (0,L),\\
&d_* b'(L,t) = d c'(L,t) = 0 \quad \text{in } (0,T).
\end{aligned} \tag{7}$$

We are concerned with the inverse problem of determining the (location, intensity) of the sources $F = (r, f(\cdot))$ and $G = (s, g(\cdot))$ when measurements are uniquely made on the DO concentration $c(\cdot,\cdot)$. Assume that ξ_G, ξ_D indicates the position of the observation stations, with $0 < \xi_G < \xi_D < L$ and that the following observations are available

$$B[F,G] = \left\{\left(c, Dc'\right)(\xi_G,\cdot), \left(c, Dc'\right)(\xi_D,\cdot), \text{ in } (0,T)\right\}. \tag{8}$$

The identifiability question may be expressed as follows: is B injective? The answer will be: Yes! To prove it, let us consider two solutions $(b_m, c_m)_{m=1,2}$ of the dispersion–reaction system related to sources (F_m, G_m). We want to show that if they provide the same observations (8), then they are equal. The ultimate objective is therefore to establish that

$$\left(B[F_1,G_1] = B[F_2,G_2]\right) \quad \text{implies that } (b_1,c_1) = (b_2,c_2).$$

We begin by stating a preparatory lemma based on Theorem 2.4. We have that

Lemma 3.1. *Suppose that $B[F_1,G_1] = B[F_2,G_2]$, then*

$$(b_1,c_1) = (b_2,c_2) \quad \textit{in } \left([0,\xi_G]\cup[\xi_D,L]\right)\times(0,T).$$

Proof. Let us focus on $[0,\xi_G]\times(0,T)$. The same argument applies as well to $[0,\xi_D]\times(0,T)$. Letting $\zeta = (b_2 - b_1)$ and $\eta = (c_2 - c_1)$. They are such that

$$\begin{aligned}
&\partial_t \zeta - \left(d_* \zeta'\right)' + r_* \zeta = 0 \quad \text{in } [0,\xi_G]\times(0,T),\\
&\partial_t \eta - \left(d \eta'\right)' + r\eta - r_* \zeta = 0 \quad \text{in } [0,\xi_G]\times(0,T),\\
&\zeta(x,0) = \eta(x,0) = (0,0) \quad \text{in } [0,\xi_G],\\
&\zeta(0,t) = \eta(0,t) = (0,0) \quad \text{in } (0,T),\\
&\left(\eta, d\eta'\right)(\xi_G,t) = (0,0) \quad \text{in } (0,T).
\end{aligned}$$

The last boundary conditions come from the coincidence of the observations on c_m, $m = 1,2$. This means that $(c_1, dc_1')(\xi_G,\cdot) = (c_2, dc_2')(\xi_G,\cdot)$. On account of the uniqueness established in Theorem 2.4, we derive immediately that $(\zeta,\eta) = (0,0)$ in $[0,\xi_G]\times(0,T)$. The same result holds in the portion $[\xi_D,L]\times(0,T)$. $\square$

We are now in a position to give the final result, the identifiability of the sources for the system (7).

Theorem 3.2. *Assume that* $B[F_1, G_1] = B[F_2, G_2]$*, then*

$$\big(r_1, f_1(t)\big) = \big(r_2, f_2(t)\big) \quad \textit{in } (0, T),$$
$$\big(s_1, g_1(t)\big) = \big(s_2, g_2(t)\big) \quad \textit{in } (0, T).$$

Idea of the proof. It is proceeded step by step. We start by Lemma 3.1 which allows to uncouple the identifiability issue for F and G. Then, we achieve following the lines of [1]. □

4. Extensions

The addition of advective transport to be able to consider pollution in flowing stream environments does not bring in more fundamental complications to the analysis. It makes sense to include this term to extend applicability to the wide range of flowing stream and river systems in which case the advective transport may be likely the most important. It is possible to handle moving pointwise sources at the cost of more technical work. Another extension is that engineers may be facing the detection of more than one pollution source. In [1], it is already noticed that for the scalar model (for BOD for instance), at least $2k$ observations are required to obtain the identifiability for k point-wise sources. They should be distributed in a particular way. Between each pair of neighboring sources one should place two distinct observation points. The same rule has to be observed here to achieve identifiability.

References

[1] M. Andrle, F. Ben Belgacem, A. El Badia, Identification of moving point-wise sources in an advection–dispersion–reaction equation, Inverse Problems 27 (2011) 025007.
[2] C. Bernardi, C. Canuto, Y. Maday, Generalized inf-sup condition for Chebyshev spectral approximation of the Stokes problem, SIAM J. Numer. Anal. 25 (1988) 1237–1271.
[3] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, 1991.
[4] R. Dautray, J.-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, vol. 5, Springer-Verlag, 1992.
[5] A. Pazy, Semi-groups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983, pp. 100–101.
[6] H.W. Streeter, E.B. Phelps, A study of the pollution and natural purification of the Ohio river, US Public Health Bull. 146 (1925).