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Differential Geometry

Classes of compact non-Kähler manifolds

Classes de variétés compactes non Kähleriennes

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A R T I C L E I N F O

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A B S T R A C T

We study various classes of compact non-Kähler manifolds, many of which already exist in the literature, which are characterized by positive forms and currents. The goal of the note is to present an overview that highlights the links between the various classes and raises some interesting problems.

R É S U M É

Nous étudions différentes classes de variétés compactes non Kähleriennes, dont beaucoup existent déjà dans la littérature, qui se caractérisent par des formes et des courants positifs. Le but de la note est de présenter une vue d'ensemble mettant en évidence les liens entre les différentes classes et pointant quelques problèmes intéressants.

1. Introduction

Let M be a complex manifold of dimension $n \geqslant 3$, and let p be an integer, $1 \leqslant p \leqslant n-1$. In [1] the following definitions were given:

Definition 1.1.

(a) M is a p-Kähler manifold if it has a closed transverse (i.e. strictly weakly positive) (p,p)-form Ω, which is called a p-Kähler form.
(b) M is a p-symplectic manifold if it has a closed transverse real $2p$-form Ψ; that is, $d\Psi = 0$ and $\Omega := \Psi^{p,p}$ (the (p,p)-component of Ψ) is transverse.

For $p = 1$, while a 1-Kähler manifold is simply a Kähler manifold, the 1-symplectic condition means that there is a symplectic 2-form Ψ which tames the given complex structure J (in the sense of Mc Duff and Gromov, see [11,8]; see moreover [16], pp. 249–252). We get a hermitian metric with fundamental form α (not closed, in general) defined by $2\alpha(v,w) := \Psi(v,w) + \Psi(Jv,Jw)$.

For $p = n-1$, we get a hermitian metric too, because every transverse $(n-1,n-1)$-form Ω is in fact given by $\Omega = \omega^{n-1}$, where ω is a transverse $(1,1)$-form.

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This case was studied by Michelson in [12], where $(n-1)$-Kähler manifolds are called *balanced* manifolds. Moreover, $(n-1)$-symplectic manifolds are called *strongly Gauduchon manifolds* by Popovici (see [13], Definition 3.1 and Lemma 3.2, and also [14] and [15]).

When $1 < p < n-1$, and ω is a transverse $(1,1)$-form, $d\omega^p = 0$ implies $d\omega = 0$: hence in the intermediate cases $(1 < p < n-1)$ the (p,p)-form Ω in Definition 1.1 is not of the form $\Omega = \omega^p$, in general. Therefore we will not look for "good" hermitian metrics, but will instead handle transverse forms or positive currents, as we will explain now.

When M is compact, we proved in [1] a characterization in terms of positive currents (see 2.1 and 2.3 below):

Theorem 1.2.

(a) *M is a p-Kähler manifold if and only if there is no non trivial positive current of bidimension (p,p) which is the component of a boundary.*
(b) *M is p-symplectic if and only if there is no non trivial positive current of bidimension (p,p) which is a boundary.*

The technique, which stems from the work of Sullivan [16], is based on the Hahn–Banach separation Theorem (on dual spaces of forms and currents). In particular, notice that the currents which are involved are positive in the sense of Lelong, i.e. strongly positive, so that the dual cone is that of weakly positive forms.

In fact, following [9], we can look at three different cones of "positive" (p,p)-forms and (p,p)-currents (i.e. of degree (p,p)): weakly positive ($WP^{p,p}$), positive ($P^{p,p}$) and strongly positive ($SP^{p,p}$).

Only for $p = 1$ or $p = n-1$, the cones coincide (see [9]); in the intermediate cases, for instance, while $\alpha \in P^{p,p}$ implies $\alpha^2 \in P^{2p,2p}$, this does not work in $WP^{p,p}$. More than that, the deep difference is based on the presence of negative eigenvalues in strictly weakly positive forms.

We have chosen the weaker notion of "positive" form in Definition 1.1 and Theorem 1.2 (transverse means strictly weakly positive), but it would be interesting to look also to other choices (some of this in [3]).

2. *p*-Kähler conditions

Actually, the purpose of this Note is to emphasize the links among the various conditions of closedness and exactness for forms and currents, to get a kind of hierarchy among non-Kähler manifolds. To this aim, let us introduce the following list of definitions and characterization theorems, in which we include for sake of completeness also Definition 1.1 and Theorem 1.2. As usual, M is a compact n-dimensional manifold, and $1 \leqslant p \leqslant n-1$.

Theorem 2.1. *Characterization of p-Kähler (pK) manifolds.*

M has a strictly weakly positive (i.e. transverse) (p,p)-form Ω with $\partial\Omega = 0$, if and only if M has no positive currents $T \neq 0$, of bidimension (p,p), such that $T = \partial\bar{S} + \bar{\partial}S$ for some current S of bidimension $(p,p+1)$ (i.e. T is the (p,p)-component of a boundary).

Theorem 2.2. *Characterization of weakly p-Kähler (pWK) manifolds.*

M has a strictly weakly positive (i.e. transverse) (p,p)-form Ω with $\partial\Omega = \partial\bar{\partial}\alpha$, if and only if M has no positive currents $T \neq 0$, of bidimension (p,p), such that $T = \partial\bar{S} + \bar{\partial}S$ for some current S of bidimension $(p,p+1)$ with $\partial\bar{\partial}S = 0$ (i.e. T is closed and is the (p,p)-component of a boundary).

Theorem 2.3. *Characterization of p-symplectic (pS) manifolds.*

M has a real $2p$-form $\Psi = \sum_{a+b=2p} \Psi^{a,b}$, such that $d\Psi = 0$ and the (p,p)-form $\Omega := \Psi^{p,p}$ is strictly weakly positive, if and only if M has no positive currents $T \neq 0$, of bidimension (p,p), such that $T = dR$ for some current R (i.e. T is a boundary).

Theorem 2.4. *Characterization of p-pluriclosed (pPL) manifolds.*

M has a strictly weakly positive (p,p)-form Ω with $\partial\bar{\partial}\Omega = 0$, if and only if M has no positive currents $T \neq 0$, of bidimension (p,p), such that $T = \partial\bar{\partial}A$ for some current A of bidimension $(p+1,p+1)$.

Proof. Theorem 2.1 and Theorem 2.3 are proved in [1], Theorem 1.17, using the techniques of [10] and [16]; in particular:

Theorem 2.1 for $p = 1$ is Theorem 14 in [10];

Theorem 2.1 for $p = n-1$ is Theorem 4.7 in [12];

Theorem 2.2 for $p = 1$ is proved in [10], Theorem 38; in fact, Theorem 2.2 is related to a question posed by Harwey and Lawson in their paper (Section 5 in [10]), about the use of *closed* currents in characterization theorems (this is important because closed positive currents are flat in the sense of Federer).

Theorem 2.3 for $p = 1$ is proved in [16], Theorems III.2 and III.11;

Theorem 2.3 for $p = n-1$ is proved also in [13], Proposition 3.3.

Theorem 2.4 for $p = 1$ is proved in [5], Theorem 3.3; recall also a result of Gauduchon ([7]) (for $p = n-1$), who proved that every compact n-dimensional manifold is $(n-1)PL$ (in fact, for $p = n-1$, the current A in Theorem 2.4 reduces to a plurisubharmonic global function on a compact complex manifold, hence to a constant).

The other proofs are similar to those of Theorems 2.1 and 2.3. □

When M satisfies one of these characterization theorems, in the rest of the Note we will call it generically a "p-Kähler" manifold.

Remark. As regards Theorem 2.3, let us write the condition $d\Psi = 0$ in terms of a condition on $\partial\Omega$, as in the other theorems; $d\Psi = 0$ is equivalent to:

(i) $\bar\partial\Psi^{n-j,2p-n+j} + \partial\Psi^{n-j-1,2p-n+j+1} = 0$, for $j = 0, \ldots, n-p-1$, when $n \leqslant 2p$, and
(ii) $\partial\Psi^{2p,0} = 0$, $\bar\partial\Psi^{2p-j,j} + \partial\Psi^{2p-j-1,j+1} = 0$, for $j = 0, \ldots, p-1$, when $n > 2p$.

In particular, $\partial\Omega = \partial\Psi^{p,p} = -\bar\partial\Psi^{p+1,p-1}$ (which is the only condition when $p = n-1$).
In the same manner we can express the condition $T = dR$.

Corollary 2.5. *(Hierarchy between classes of "p-Kähler" manifolds)*:

$$pK \Rightarrow pWK \Rightarrow pS \Rightarrow pPL.$$

3. Some questions

Some questions arise immediately:
Question 1: Are these classes of manifolds really distinct?
Question 2: In which cases classes coincide (that is, $pPL \Rightarrow pK$)?
Question 3: What kind of results naturally extend from the pK-case (the most studied) to the other cases of "p-Kähler" manifolds?

We shall give some partial answer, starting from Question 3, which is indeed only an indication of possible questions (see f.i. [14], Introduction).

First of all, let us state a couple of quite obvious remarks:

(a) Let M be a "p-Kähler" manifold, and let $f : S \to M$ be a holomorphic immersion, with $\dim S > p$. Then S is a "p-Kähler" manifold too.
(b) Let M be an n-dimensional "p-Kähler" manifold, and let $f : M \to P$ be a holomorphic submersion, with $\dim P = n' > n-p$. Then P is "$(n'-n+p)$-Kähler".

The following results give an example of properties which can be extended, keeping more or less the same proof for all type of "p-Kähler" manifolds (see also [5] for $p = 1$):

Proposition 3.1. *Let M be an n-dimensional "p-Kähler" manifold. Then there are no non trivial exact simple holomorphic $(n-p)$-forms on M.*

Proof. Suppose α is an exact simple holomorphic $(n-p)$-form; hence $\alpha = \partial\beta$, and, for a suitable constant σ_{n-p}, $T := \sigma_{n-p}\alpha \wedge \bar\alpha$ is a positive $\partial\bar\partial$-exact current. Thus $T = 0$, i.e. $\alpha = 0$. □

Proposition 3.2. *(See [10], Theorem 17 and [5], Theorem 4.5.) Suppose $f : M \to P$ is a holomorphic submersion with 1-dimensional fibres onto a "1-Kähler" manifold P. Then M is "1-Kähler" if and only if the condition required on currents in the characterization theorems holds on the generic fibre.*

A deeper result is Theorem 1.3 in [14], where a result about modifications is extended from the $(n-1)K$-case to the $(n-1)S$-case.

As for Question 2, we proved in [3] that for holomorphically parallelizable complex manifolds, all classes coincide, for every p; for $p = 1$, the classes coincide on Moishezon manifolds ([5], Theorem 7.4).

Moreover, let us recall a couple of classical results. In [4], Lemma 5.15, the $\partial\bar\partial$-Lemma is stated in the general case of a bounded double complex of vector spaces $(K^{*,*}, \partial, \bar\partial)$, with associated simple complex (K, d). Using this notation for the de Rham complex of forms or currents, we say that a compact complex manifold M satisfies the $\partial\bar\partial$-Lemma if for every m, in K^m we have

$$\operatorname{Ker}\bar\partial \cap \operatorname{Im}\partial = \operatorname{Im}\partial\bar\partial = \operatorname{Ker}\partial \cap \operatorname{Im}\bar\partial,$$

or equivalently in K^{m-1} we have

$$\operatorname{Im}\bar\partial + \operatorname{Ker}\partial = \operatorname{Ker}\partial\bar\partial = \operatorname{Im}\partial + \operatorname{Ker}\bar\partial.$$

On the other side, in the same context but using the formalism of cohomology groups introduced in [17], we have $\forall p, q$ two exact sequences, where the maps are induced by the identity:

$$0 \to A^{p,q} := \frac{\operatorname{Im}\bar\partial \cap \operatorname{Im}\partial}{\operatorname{Im}\partial\bar\partial} \to B^{p,q} := \frac{\operatorname{Ker}\bar\partial \cap \operatorname{Im}\partial}{\operatorname{Im}\partial\bar\partial} \to H^{p,q}_{\bar\partial} \to V^{p,q} \to C^{p,q} := \frac{\operatorname{Ker}\partial\bar\partial}{\operatorname{Ker}\bar\partial + \operatorname{Im}\partial} \to 0,$$

$$0 \to D^{p,q} := \frac{\operatorname{Im}\bar\partial \cap \operatorname{Ker}\partial}{\operatorname{Im}\partial\bar\partial} \to \Lambda^{p,q} \to H^{p,q}_{\bar\partial} \to E^{p,q} := \frac{\operatorname{Ker}\partial\bar\partial}{\operatorname{Ker}\partial + \operatorname{Im}\bar\partial} \to F^{p,q} := \frac{\operatorname{Ker}\partial\bar\partial}{\operatorname{Ker}\bar\partial + \operatorname{Ker}\partial} \to 0,$$

where

$$H^{p,q}_{\bar\partial} = \frac{\operatorname{Ker}\bar\partial}{\operatorname{Im}\bar\partial}, \qquad V^{p,q} := \frac{\operatorname{Ker}\partial\bar\partial}{\operatorname{Im}\bar\partial + \operatorname{Im}\partial}, \qquad \Lambda^{p,q} := \frac{\operatorname{Ker}\bar\partial \cap \operatorname{Ker}\partial}{\operatorname{Im}\partial\bar\partial},$$

and moreover there are the following isomorphisms: $V^{p,q} = V^{q,p}$, $\Lambda^{p,q} = \Lambda^{q,p}$, $A^{p,q} = A^{q,p}$, $F^{p,q} = F^{q,p}$, $D^{p,q} = B^{q,p}$, $E^{p,q} = C^{q,p}$, induced by the conjugation, and $C^{p,q} = D^{p,q+1}$, $E^{p,q} = B^{p+1,q}$, induced by $\bar\partial$ and ∂.

In [17], M is called a *regular* manifold when the maps $\Lambda^{p,q} \to H^{p,q}_{\bar\partial}$ are injective $\forall p, q$ (as a matter of fact, this implies that the maps are isomorphisms). By easy computations, we get:

Theorem 3.3. *The following statements are equivalent, for a compact manifold M:*

(i) *$\forall p, q$, for every closed $T \in K^{p,q}$, T is equivalently in* $\operatorname{Im} d$, $\operatorname{Im}\partial$, $\operatorname{Im}\bar\partial$, $\operatorname{Im}\partial\bar\partial$.
(ii) *M satisfies the $\partial\bar\partial$-Lemma.*
(iii) *M is a regular manifold.*

Corollary 3.4. *On a regular manifold, $\forall p$, $pWK = pS = pPL$.*

Proof. Notice that pPL implies pWK, because if $T = \partial\bar S + \bar\partial S$ with $\partial\bar\partial S = 0$, then $\partial\bar S \in \operatorname{Ker}\bar\partial \cap \operatorname{Im}\partial = \operatorname{Im}\partial\bar\partial$, so that $\partial\bar S = \partial\bar\partial B$ and $T = \partial\bar S + \bar\partial S = \partial\bar\partial(B - \bar B)$. The examples below show that the hypothesis is necessary. □

Corollary 3.5. *Every regular manifold is $(n-1)WK$, since $(n-1)WK = (n-1)PL$.*

Of course, the $\partial\bar\partial$-Lemma is not necessary to ensure the previous results: in fact, if $B^{p,p} = 0$, then $pWK = pS = pPL$ (as a matter of fact, it would be sufficient to have $B^{p,p} \cap \{T^{p,p} \geqslant 0\} = 0$).

Fu and Yau introduce in [6] a weak form of the $\partial\bar\partial$-Lemma ([6], Definition 5; compare it to our condition $A^{n-1,n} = 0$ which is verified when $H^{n-2,n}_{\bar\partial} = 0$); they prove that small deformations $\{X_t\}$ of a balanced manifold are balanced, when the manifolds X_t satisfy this weakened $\partial\bar\partial$-Lemma (Theorem 6). In our setting, what they need is in fact the following statement: *When Ω is a $(n-1)$-symplectic form, then it is also a $(n-1)$-WK form*; this holds when $A^{n-1,n} = 0$.

Finally, as regards Question 1, in [2] we exhibited a small deformation $I_{3,t}$ of the Iwasawa manifold I_3, that is not $2K$ (i.e., not balanced); $I_{3,t}$ is obviously $2PL$, and a simple computation shows that it is $2S$ but not $2WK$. Similar examples can be given using solvmanifolds, instead of the nilmanifold I_3.

In dimension bigger than 3, that is, in the non trivial p-Kähler case, we constructed in [2] a small deformation $\eta\beta_{5,t}$ of the 5-dimensional nilmanifold $\eta\beta_5$ (which is not $1K$ and $2K$, but is $3K$ and $4K$), which is not $3K$ nor $3WK$, but is "4-Kähler".

Up to now, the difference between p-PL and p-S is clear at least when $p = n-1$; p-S and p-WK are distinguished by $I_{3,t}$, and the difference between p-WK and p-K is suggested by Corollary 3.4.

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