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# Localising subcategories for cochains on the classifying space of a finite group ☆

## *Sous-catégories localisantes pour les co-chaînes des espaces classifiants de groupes finis*

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ABSTRACT

The localising subcategories of the derived category of the cochains on the classifying space of a finite group are classified. They are in one to one correspondence with the subsets of the set of homogeneous prime ideals of the cohomology ring $H^*(G,k)$.



RÉSUMÉ

Nous identifions les sous-catégories localisantes de la catégorie dérivée des co-chaînes, à coefficients dans un corps $k$ de caractéristique $p$, sur l'espace classifiant d'un groupe fini $G$. Elles sont en correspondance biunivoque avec les sous-ensembles de l'ensemble des idéaux premiers homogènes de l'anneau de co-homologie $H^*(G,k)$.



### 1. Introduction

Let $G$ be a finite group and $k$ a field of characteristic $p$. Let $C^*(BG;k)$ be the cochains on the classifying space $BG$. Using the machinery of Elmendorf, Kříž, Mandell and May [8], one can regard $C^*(BG;k)$ as a strictly commutative $S$-algebra over the field $k$. The derived category $\mathsf{D}(C^*(BG;k))$ has thus a structure of a tensor triangulated category via the left derived tensor product $-\otimes^{\mathbf{L}}_{C^*(BG;k)}-$. The unit for the tensor product is $C^*(BG;k)$.

In this paper we apply techniques and results from [3–6] to classify the localising subcategories of $\mathsf{D}(C^*(BG;k))$. More precisely, there is a notion of stratification for triangulated categories via the action of a graded commutative ring which implies that the localising subcategories are parameterised by sets of homogeneous prime ideals [4]. For $\mathsf{D}(C^*(BG;k))$ we use the natural action of the endomorphism ring of the tensor identity which is isomorphic to the cohomology algebra $H^*(G,k)$ of the group $G$.

**Theorem 1.1.** *The derived category $\mathsf{D}(C^*(BG;k))$ is stratified by the ring $H^*(G,k)$. This yields a one to one correspondence between the localising subcategories of $\mathsf{D}(C^*(BG;k))$ and subsets of the set of homogeneous prime ideals of $H^*(G,k)$.*

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It is proved in [6] that there is an equivalence of tensor triangulated categories between $\mathsf{D}(C^*(BG;k))$ and the localising subcategory of $\mathsf{K}(\mathsf{Inj}\,kG)$ generated by the tensor identity. Here, $\mathsf{K}(\mathsf{Inj}\,kG)$ is the homotopy category of complexes of injective (= projective) $kG$-modules, studied in [6,9].

The main theorem of [5] states that $\mathsf{K}(\mathsf{Inj}\,kG)$ is stratified as a tensor triangulated category by $H^*(G,k)$. Theorem 1.1 is a consequence of a more general result concerning tensor triangulated categories, which is described below.

Let $(\mathsf{T},\otimes,\mathbb{1})$ be a compactly generated tensor triangulated category, as described in [3, §8], and $R$ a graded commutative Noetherian ring acting on $\mathsf{T}$ via a homomorphism $R \to \operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$. In this case, for each homogeneous prime ideal $\mathfrak{p}$ of $R$ there exists a *local cohomology functor* $\varGamma_{\mathfrak{p}}\colon \mathsf{T} \to \mathsf{T}$; see [3]. The *support* of an object $X$ in $\mathsf{T}$ is then defined to be

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}} X \neq 0\}.$$

The condition that $\mathsf{T}$ is *stratified* by the action of $R$ means that assigning a subcategory $\mathsf{S}$ of $\mathsf{T}$ to its support

$$\operatorname{supp}_R \mathsf{S} = \bigcup_{X\in\mathsf{S}} \operatorname{supp}_R X$$

yields a bijection between *tensor ideal* localising subcategories of $\mathsf{T}$ and subsets of the homogeneous prime ideal spectrum $\operatorname{Spec} R$ contained in $\operatorname{supp}_R \mathsf{T}$; see [4, Theorem 4.2]. Theorem 1.1 is thus a special case of the result below that relates tensor ideal localising subcategories of $\mathsf{T}$ and the localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, the localising subcategory of $\mathsf{T}$ generated by the tensor unit. We note that $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ is a compactly generated tensor triangulated category in its own right and that $R$ acts on it as well.

**Theorem 1.2.** *Suppose that the Krull dimension of $R$ is finite. If $\mathsf{T}$ is stratified by $R$ as a tensor triangulated category, then so is $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, and there is a bijection*

$$\left\{\begin{matrix}\textit{Tensor ideal localising}\\ \textit{subcategories of}\ \mathsf{T}\end{matrix}\right\} \xrightarrow{\ \sim\ } \left\{\begin{matrix}\textit{Localising subcategories}\\ \textit{of}\ \operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\end{matrix}\right\}.$$

*It assigns each tensor ideal localising subcategory $\mathsf{S}$ of $\mathsf{T}$ to $\mathsf{S}\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$.*

**Remark 1.3.** The theorem is not true without the assumption that $\mathsf{T}$ is stratified by $R$. For example, let $\mathsf{T}$ be the derived category of quasi-coherent sheaves on the projective line $\mathbb{P}^1_k$. The tensor unit is $\mathcal{O}$. In this example there are no proper localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathcal{O})$ since $\operatorname{End}^*_{\mathsf{T}}(\mathcal{O}) = k$, while there are many tensor ideal localising subcategories of $\mathsf{T}$.

**Remark 1.4.** The assumption that the Krull dimension of $R$ is finite is artificial, and is used only to ensure that for each $X \in \mathsf{T}$ and $\mathfrak{p} \in \operatorname{Spec} R$ the object $\varGamma_{\mathfrak{p}} X$ belongs to $\operatorname{Loc}_{\mathsf{T}}(X)$. One can replace this condition by, for instance, the assumption that $\mathsf{T}$ arises as the homotopy category of a Quillen model category [10, §6].

## 2. Localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$

In this section $\mathsf{T}$ is a triangulated category with set-indexed coproducts and the tensor product $\otimes$ provides a symmetric monoidal structure with unit $\mathbb{1}$ on $\mathsf{T}$, which is exact in each variable and preserves set-indexed coproducts.

The proof of Theorem 1.2 is based on a sequence of elementary lemmas. The first one describes the tensor ideal localising subcategory of $\mathsf{T}$ which is generated by a class $\mathsf{C}$ of objects; we denote this by $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{C})$.

**Lemma 2.1.** *Let $\mathsf{C}$ be a class of objects of $\mathsf{T}$. Then*

$$\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{C}) = \operatorname{Loc}_{\mathsf{T}}\big(\{X\otimes Y \mid X\in\mathsf{C},\ Y\in\mathsf{T}\}\big).$$

**Proof.** Set $\mathsf{S} = \operatorname{Loc}_{\mathsf{T}}(\{X\otimes Y \mid X\in\mathsf{C},\ Y\in\mathsf{T}\})$. It suffices to show that $\mathsf{S}$ is tensor ideal. This means that $F\mathsf{S}\subseteq\mathsf{S}$ for each tensor functor $F = -\otimes Y$, which is an immediate consequence of Lemma 2.2 below. □

**Lemma 2.2.** *Let $F\colon\mathsf{U}\to\mathsf{V}$ be an exact functor between triangulated categories that preserves set-indexed coproducts. If $\mathsf{C}$ is a class of objects of $\mathsf{U}$, then*

$$F\operatorname{Loc}_{\mathsf{U}}(\mathsf{C}) \subseteq \operatorname{Loc}_{\mathsf{V}}(F\mathsf{C}).$$

**Proof.** The preimage $F^{-1}\operatorname{Loc}_{\mathsf{V}}(F\mathsf{C})$ is a localising subcategory of $\mathsf{U}$ containing $\mathsf{C}$. Thus it contains $\operatorname{Loc}_{\mathsf{U}}(\mathsf{C})$, and one gets

$$F\operatorname{Loc}_{\mathsf{U}}(\mathsf{C}) \subseteq FF^{-1}\operatorname{Loc}_{\mathsf{V}}(F\mathsf{C}) \subseteq \operatorname{Loc}_{\mathsf{V}}(F\mathsf{C}). \qquad \square$$

**Lemma 2.3.** *Let $\Gamma\colon \mathsf{T}\to\mathsf{T}$ be a colocalisation functor that preserves set-indexed coproducts. Then for any $X\in\mathsf{T}$ and $Y\in\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, there is a natural isomorphism*

$$\Gamma X\otimes Y \xrightarrow{\sim} \Gamma(X\otimes Y).$$

**Remark 2.4.** There is an analogous result for a localisation functor $L\colon\mathsf{T}\to\mathsf{T}$ that preserves set-indexed coproducts: For any $X\in\mathsf{T}$ and $Y\in\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, there is a natural isomorphism $L(X\otimes Y)\xrightarrow{\sim} LX\otimes Y$.

**Proof.** A colocalisation functor $\Gamma$ comes with a natural morphism $\Gamma X\to X$. Tensoring this with an object $Y\in\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ gives a morphism $\Gamma X\otimes Y\to X\otimes Y$ that factors through the natural morphism $\Gamma(X\otimes Y)\to X\otimes Y$. Here, one uses that $\Gamma X\otimes Y$ belongs to $\Gamma\mathsf{T}$, since the objects $Y'\in\mathsf{T}$ with $\Gamma X\otimes Y'\in\Gamma\mathsf{T}$ form a localising subcategory containing $\mathbb{1}$. The induced morphism $\phi_Y\colon\Gamma X\otimes Y\to\Gamma(X\otimes Y)$ is an isomorphism. To see this, observe that the objects $Y'\in\mathsf{T}$ such that $\phi_{Y'}$ is an isomorphism form a localising subcategory containing $\mathbb{1}$. □

**Proposition 2.5.** *Suppose that the unit $\mathbb{1}$ is compact in $\mathsf{T}$ and let $\Gamma\colon\mathsf{T}\to\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ denote the right adjoint of the inclusion $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\to\mathsf{T}$. If $\mathsf{S}$ is a localising subcategory of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, then*

$$\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})=\Gamma\big(\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})\big)=\mathsf{S}.$$

**Proof.** We verify each of the following inclusions

$$\mathsf{S}\subseteq\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\subseteq\Gamma\big(\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})\big)\subseteq\mathsf{S}.$$

The first one is clear. Composing the functor $\Gamma$ with the inclusion $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\to\mathsf{T}$ yields a colocalisation functor that preserves set-indexed coproducts, since $\mathbb{1}$ is compact. For an object $X$ in $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, we have $\Gamma X\cong X$. This gives the second inclusion. Applying Lemma 2.3 together with the description of $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})$ from Lemma 2.1 yields the third inclusion. □

**Corollary 2.6.** *Suppose that the unit $\mathbb{1}$ is a compact object in $\mathsf{T}$. Assigning each localising subcategory $\mathsf{S}$ of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ to $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})$ gives a bijection*

$$\left\{\begin{matrix}\text{Localising subcategories}\\ \text{of }\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\end{matrix}\right\}\xrightarrow{\sim}\left\{\begin{matrix}\text{Tensor ideal localising subcategories of}\\ \mathsf{T}\text{ generated by objects from }\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\end{matrix}\right\}.$$

**Proof.** The inverse map sends $\mathsf{U}\subseteq\mathsf{T}$ to $\mathsf{U}\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$. □

We are now ready to prove Theorem 1.2. Note that in this $\mathsf{T}$ is a compactly generated tensor triangulated category, which entails a host of additional requirements; see [3, §8] for a list.

**Proof of Theorem 1.2.** It follows from Proposition 2.5 that the assignment

$$\mathsf{S}\longmapsto\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})$$

is an injective map from the localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ to the tensor ideal localising subcategories of $\mathsf{T}$. In general, it is not bijective, as the example of Remark 1.3 shows. However, since $\mathsf{T}$ is stratified by $R$ as a tensor triangulated category, it follows from [4, §7] that each tensor ideal localising subcategory is generated by a set of objects of the form $\Gamma_{\mathfrak{p}}\mathbb{1}$. Since $R$ has finite Krull dimension, [4, Theorem 3.4] yields that $\Gamma_{\mathfrak{p}}\mathbb{1}$ is in $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$. Therefore, given a tensor ideal localising subcategory $\mathsf{U}$ of $\mathsf{T}$, the localising subcategory

$$\mathsf{U}'=\operatorname{Loc}_{\mathsf{T}}\big(\{\Gamma_{\mathfrak{p}}\mathbb{1}\mid\mathfrak{p}\in\operatorname{Supp}_R\mathsf{U}\}\big)\subseteq\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$$

satisfies $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{U}')=\mathsf{U}$. This proves the surjectivity of the assignment. Moreover, we have shown that each localising subcategory of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ is generated by objects of the form $\Gamma_{\mathfrak{p}}\mathbb{1}$, so $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ is stratified by the action of $R$; see [4, Theorem 4.2]. □

## 3. The cohomological nucleus

Let $(\mathsf{T},\otimes,\mathbb{1})$ be a compactly generated tensor triangulated category and let $R$ be a graded commutative Noetherian ring acting on $\mathsf{T}$ via a homomorphism $R\to\operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$. Suppose in addition that $R$ has finite Krull dimension.

We define the *cohomological nucleus* of $\mathsf{T}$ as the set of homogeneous prime ideals $\mathfrak{p}$ of $R$ such that there exists an object $X\in\mathsf{T}$ satisfying $\operatorname{Hom}^*_{\mathsf{T}}(\mathbb{1},X)=0$ and $\Gamma_{\mathfrak{p}}X\neq 0$. This definition is motivated by work of Benson, Carlson, and Robinson in the context of modular group representations [2].

For $\mathfrak{p}$ in $\operatorname{Spec}R$ consider the tensor ideal localising subcategory

$$\Gamma_{\mathfrak{p}}\mathsf{T} = \{Y \in \mathsf{T} \mid Y \cong \Gamma_{\mathfrak{p}} X \text{ for some } X \in \mathsf{T}\}.$$

Note that an object $X \in \mathsf{T}$ belongs to $\Gamma_{\mathfrak{p}}\mathsf{T}$ if and only if $\mathrm{Hom}^*_{\mathsf{T}}(C, X)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion for every compact $C \in \mathsf{T}$, by [3, Corollary 4.10]. The result below gives a local description of the cohomological nucleus.

**Proposition 3.1.** *Let* $\mathfrak{p}$ *be a homogeneous prime ideal of* $R$. *The following conditions are equivalent*:

(1) *Every object* $X$ *in* $\mathsf{T}$ *with* $\mathrm{Hom}^*_{\mathsf{T}}(\mathbb{1}, X) = 0$ *satisfies* $\Gamma_{\mathfrak{p}} X = 0$.
(2) *One has* $\mathrm{Loc}_{\mathsf{T}}(\Gamma_{\mathfrak{p}}\mathbb{1}) = \Gamma_{\mathfrak{p}}\mathsf{T}$.
(3) *Every localising subcategory of* $\Gamma_{\mathfrak{p}}\mathsf{T}$ *is a tensor ideal of* $\mathsf{T}$.

**Proof.** The Krull dimension of $R$ is finite, so $\Gamma_{\mathfrak{p}} X$ is in $\mathrm{Loc}_{\mathsf{T}}(X)$ for each $X$ in $\mathsf{T}$, by [4, Theorem 3.4]. This fact is used without further comment.

(1) $\Rightarrow$ (2): Set $\mathsf{S} = \mathrm{Loc}_{\mathsf{T}}(\Gamma_{\mathfrak{p}}\mathbb{1})$. Note that $\mathsf{S} \subseteq \Gamma_{\mathfrak{p}}\mathsf{T}$; we claim that equality holds. Indeed, $\mathsf{S} \subseteq \mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$ and also $\mathrm{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S}) = \Gamma_{\mathfrak{p}}\mathsf{T}$, since $\Gamma_{\mathfrak{p}} = \Gamma_{\mathfrak{p}}\mathbb{1} \otimes -$. Thus, for any $X$ in $\Gamma_{\mathfrak{p}}\mathsf{T}$ from Proposition 2.5 one gets an exact triangle $\Gamma X \to X \to X' \to$ with $\Gamma X \in \mathsf{S}$ and $\mathrm{Hom}^*_{\mathsf{T}}(\mathbb{1}, X) = 0$. Then (1) implies $X' = 0$ and hence $X \in \mathsf{S}$.

(2) $\Rightarrow$ (3): Let $\mathsf{S}$ be a localising subcategory of $\Gamma_{\mathfrak{p}}\mathsf{T}$. Using (2) and the fact that $\Gamma_{\mathfrak{p}}\mathsf{T}$ is a tensor ideal of $\mathsf{T}$, one has $\mathrm{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S}) \subseteq \mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$. Then it follows, again from Proposition 2.5, that $\mathsf{S}$ is a tensor ideal of $\mathsf{T}$.

(3) $\Rightarrow$ (1): Assume $\mathrm{Hom}^*_{\mathsf{T}}(\mathbb{1}, X) = 0$; then $\mathrm{Hom}^*_{\mathsf{T}}(\mathbb{1}, \Gamma_{\mathfrak{p}} X) = 0$, as $\mathbb{1}$ is compact. Condition (3) implies that $\mathrm{Loc}_{\mathsf{T}}(\Gamma_{\mathfrak{p}}\mathbb{1}) = \Gamma_{\mathfrak{p}}\mathsf{T}$. Thus $\Gamma_{\mathfrak{p}} X$ belongs to $\mathrm{Loc}_{\mathsf{T}}(\Gamma_{\mathfrak{p}}\mathbb{1})$ and therefore also to $\mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$. So one obtains $\mathrm{Hom}^*_{\mathsf{T}}(\Gamma_{\mathfrak{p}} X, \Gamma_{\mathfrak{p}} X) = 0$, which implies $\Gamma_{\mathfrak{p}} X = 0$. $\square$

Consider as an example for $\mathsf{T}$ the stable module category $\mathsf{StMod}\,kG$ of a finite group $G$ with the canonical action of $R = H^*(G, k)$. We refer to [1,2] for the discussion of two variations of the nucleus, namely the *group theoretic* and the *representation theoretic* nucleus. There it is shown that $\mathrm{Loc}_{\mathsf{T}}(\mathbb{1}) = \mathsf{T}$ if and only if the centraliser of every element of order $p$ in $G$ is $p$-nilpotent and every block is either principal or semisimple, where $p$ denotes the characteristic of the field $k$.

It is convenient to define for any class $\mathsf{C}$ of objects of $\mathsf{T}$

$$\mathsf{C}^{\perp} = \{Y \in \mathsf{T} \mid \mathrm{Hom}^*_{\mathsf{T}}(X, Y) = 0 \text{ for all } X \in \mathsf{C}\},$$
$${}^{\perp}\mathsf{C} = \{X \in \mathsf{T} \mid \mathrm{Hom}^*_{\mathsf{T}}(X, Y) = 0 \text{ for all } Y \in \mathsf{C}\}.$$

Now let $\mathsf{S} = \mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$. The *representation theoretic nucleus* is by definition

$$\bigcup_{X \in \mathsf{S}^{\perp} \cap \mathsf{T}^c} \mathrm{supp}_R X.$$

Clearly, this is contained in the cohomological nucleus. It is a remarkable fact that the representation theoretic nucleus is non-empty if $\mathsf{S}^{\perp} \neq 0$; this is proved in [1,2]. Moreover, Question 13 of [7] asks whether $\mathsf{S} = {}^{\perp}(\mathsf{S}^{\perp} \cap \mathsf{T}^c)$. Note that $\mathsf{S} = {}^{\perp}(\mathsf{S}^{\perp})$ follows from general principles.

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