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Probability Theory

McKean–Vlasov diffusions: From the asynchronization to the synchronization ☆

Diffusions de McKean–Vlasov : De l'asynchronisation à la synchronisation

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ABSTRACT

We make the asymptotic analysis of the unique symmetric stationary measure of a self-stabilizing process in the small-noise limit. It has been proved in previous works that this measure converges with a linear rate in the asynchronized case and in the strictly synchronized case but it is slower in the intermediate case. The aim of this Note is to zoom around this phase transition.

RÉSUMÉ

On procède à l'analyse asymptotique de l'unique mesure stationnaire symétrique pour un processus auto-stabilisant à petit bruit. Il a été prouvé dans des travaux antécédents que cette mesure converge avec un taux linéaire dans le cas asynchrone et dans le cas strictement synchrone mais la convergence est moins rapide dans le cas intermédiaire. Le but de cette Note est de zoomer autour de cette transition de phase.

1. Introduction

We investigate the asymptotic behavior in the small-noise limit of the stationary measures of the following so-called self-stabilizing process:

$$\begin{cases} X_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t V'(X_s)\,\mathrm{d}s - \int_0^t F' * u_s(X_s)\,\mathrm{d}s \\ u_s = \mathcal{L}(X_s) \end{cases} . \tag{I}$$

Here, $*$ denotes the convolution. The motion of the process is generated by three forces. The first one is the derivative of a potential V. The second influence is a Brownian motion $(B_t)_{t\in\mathbb{R}_+}$. The third term – the so-called self-stabilizing term – represents the attraction between all the trajectories. We will assume that V and F are polynomial functions.

There is a unique strong solution for (I) if $\mathbb{E}[X_0^{8q^2}] < \infty$ with $2q = \max\{\deg(V); \deg(F)\}$, see [2]. The existence and the non-uniqueness of the stationary measures have been studied in [3,7]. In particular, under simple assumptions, if V and F are even, there is at least one symmetric stationary measure u_ϵ. The convergence in long-time towards one of the stationary

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measure(s) has been proved in [6]. The asymptotic analysis of the measure(s) u_ϵ has been made in [4,5]. Especially, the convergence has been the subject of [4] and the rate of convergence has been studied in [5].

Under the convexity of V'', F and F'' and the non-convexity of V, if we consider u_ϵ (whose limit is u_0), according to Theorem 1.2, Theorem 1.3 and Theorem 1.4 in [5], we obtain: for any $f \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ with polynomial growth,

$$\lim_{\epsilon\to 0}\frac{\log|\langle f,u_\epsilon\rangle - \langle f,u_0\rangle|}{\log\epsilon} = \Lambda(\alpha) := \begin{cases} 1 & \text{if } \alpha\neq\theta \\ 1/m_0 & \text{if } \alpha=\theta \end{cases}, \tag{II}$$

where $m_0 := \min\{k\geqslant 2 \mid V^{(2k)}(0)+F^{(2k)}(0)>0 \text{ or } F^{(2k)}(0)>0\}$, $\alpha := F''(0)$ and $\theta := -V''(0)$.

This points out a discontinuity around the phase transition $\alpha=\theta$ between the asynchronization ($V''(0)+F''(0)<0$) and the synchronization ($V''(0)+F''(0)\geqslant 0$). The term "synchronization" means that the interaction is so strong that all the stationary measures are pure states (Dirac measures) in the small-noise limit.

The aim of this Note is to prove that we can observe intermediate rates of convergence around the condition $\alpha=\theta$ provided that the potential function F depends on the small parameter ϵ. This is based on a particular example, nevertheless extensions to general situations can easily be proved.

In order to produce intermediate rates, we construct some particular model. First we choose the reference environment (resp. interaction) function $V(x) := \frac{x^4}{4}-\frac{x^2}{2}$ (resp. $F(x) := \frac{x^4}{4}+\frac{x^2}{2}$). Obviously $\theta=\alpha=1$. The example is based on a small perturbation of this reference: we consider the association between V and one of the following interaction potentials

$$F_\rho^+(x) := \frac{x^4}{4}+\frac{(1+\rho\epsilon^\eta)x^2}{2} \quad\text{or}\quad F_\rho^-(x) := \frac{x^4}{4}+\frac{(1-\rho\epsilon^\eta)x^2}{2},$$

with $\rho>0$. In other words, Eq. (I) in [5] leads to the study of symmetric invariant measures of the following self-stabilizing process:

$$\mathrm{d}X_t = \sqrt{\epsilon}\,\mathrm{d}B_t - \left\{2X_t^3 + \left(3\mathbb{E}\left[X_t^2\right] \pm \rho\epsilon^\eta\right)X_t\right\}\mathrm{d}t. \tag{III}$$

We remove the odd moments appearing in the equation since we focus our attention to symmetric laws.

We will see that, in this new timescale, the function Λ is continuous in the right part of the diopter.

The Note is organized as follows. After presenting and justifying succinctly the general results (existence of a solution, existence of a unique symmetric stationary measure u_ϵ and the convergence towards δ_0 with $\epsilon\to 0$), we will give the value of $\lim_{\epsilon\to 0}\log m_2(\epsilon)/\log\epsilon$ – where $m_2(\epsilon)$ is the second moment of u_ϵ – by using some integrals which already appeared in [1]. Then, the prefactor will be provided.

2. Preliminaries

There exists a unique strong solution for (III) (see [2]) and a unique symmetric invariant measure (see Section 4.2 in [3]) denoted by $u_\epsilon^\pm$ because we do not use ϵ in these proofs. We define $m_2^\pm(\epsilon) := \int_\mathbb{R} x^2 u_\epsilon^\pm(x)\,\mathrm{d}x$. By Section 4.2 in [3], the second moment $m_2^\pm(\epsilon)$ satisfies

$$m_2^\pm(\epsilon) = \frac{\int_{\mathbb{R}_+} x^2\exp[-\frac{1}{\epsilon}((3m_2^\pm(\epsilon)\pm\rho\epsilon^\eta)x^2+x^4)]\,\mathrm{d}x}{\int_{\mathbb{R}_+}\exp[-\frac{1}{\epsilon}((3m_2^\pm(\epsilon)\pm\rho\epsilon^\eta)x^2+x^4)]\,\mathrm{d}x}. \tag{IV}$$

Proposition 1. *For all $\eta>0$ and $\rho>0$, the sequence of symmetric invariant measures $(u_\epsilon^\pm, \epsilon>0)$ converges weakly towards δ_0.*

Proof. Let us assume the existence of a positive constant C and a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ converging towards 0 such that $m_2^\pm(\epsilon_k)>C$ for all $k\in\mathbb{N}$ (for notational simplicity, we shall drop the index k). We apply the following change of variable $x := \sqrt{\epsilon}y$ to (IV) and obtain $m_2^\pm(\epsilon) = \epsilon\xi(m_2^\pm(\epsilon)\pm\rho\epsilon^\eta/3,\epsilon)$ where

$$\xi(u,v) := \int_{\mathbb{R}_+} y^2\nu_{u,v}(y)\,\mathrm{d}y \quad\text{and}\quad \nu_{u,v}(y) := \frac{\exp[-3uy^2-vy^4]}{\int_\mathbb{R}\exp[-3uz^2-vz^4]\,\mathrm{d}z}. \tag{V}$$

Let $u\in\mathbb{R}$, $v>0$. By Jensen's inequality, $\frac{\partial\xi}{\partial u}(u,v)<0$ and $\frac{\partial\xi}{\partial v}(u,v)<0$. Since $3m_2(\epsilon)\pm\rho\epsilon^\eta>C$, for ϵ small enough, we deduce that $m_2^\pm(\epsilon)\leqslant\epsilon\,\xi(C/3,0)$. The r.h.s tends towards 0 when ϵ goes to 0 which is a nonsense because $m_2^\pm(\epsilon)>C$. $\square$

3. Main results

We shall just estimate the convergence of $m_2^\pm(\epsilon)$ as $\epsilon\to 0$ since $\langle f,u_\epsilon^\pm\rangle - \langle f,u_0^\pm\rangle$ is directly linked to $m_2^\pm(\epsilon)$. Indeed: $\langle f;u_\epsilon^\pm\rangle - \langle f;u_0^\pm\rangle = f''(0)m_2^\pm(\epsilon)+O\{m_4^\pm(\epsilon)\}$. And, $m_4^\pm(\epsilon) = o\{m_2^\pm(\epsilon)\}$ since $u_\epsilon^\pm$ converges towards δ_0.

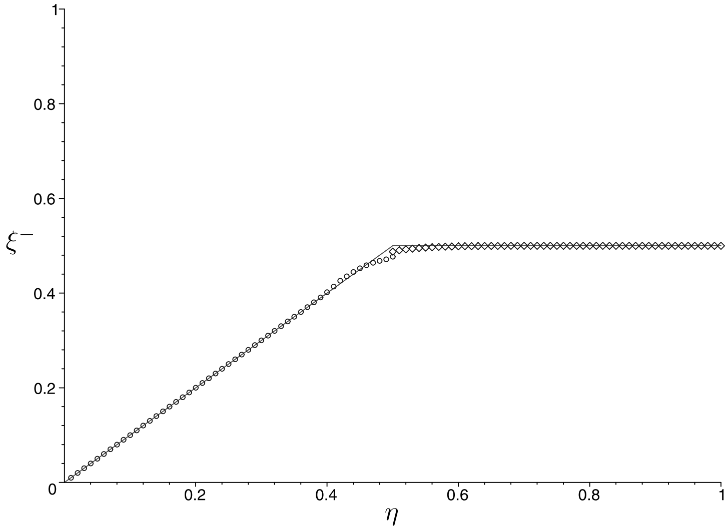

Fig. 1. Description of m_2^-.

Proposition 2. *Let $\eta > 0$ and $\rho > 0$. Then* $\lim_{\epsilon\to 0} \frac{\log m_2^+(\epsilon)}{\log \epsilon} = \Lambda_+(\eta) := 1 - \min\{\eta; \frac{1}{2}\}$.

Proof. Let us recall that $m_2^+(\epsilon) = \epsilon\xi(m_2^+(\epsilon) + \rho\epsilon^\eta/3, \epsilon)$ where ξ is defined by (V). According to the proof of Proposition 1, the function ξ is decreasing with respect to both variables. Therefore we can compute some upper-bound of $m_2^+(\epsilon)$ just by noting that $m_2^+(\epsilon) + \rho\epsilon^\eta/3 \geqslant \max\{m_2^+(\epsilon);\ \rho\epsilon^\eta/3\}$.

- The first inequality leads to $m_2^+(\epsilon) \leqslant \epsilon\xi(m_2^+(\epsilon), 0)$. The r.h.s can be computed by some change of variable: there exists some constant $c_0 > 0$ such that $m_2^+(\epsilon) \leqslant c_0\sqrt{\epsilon}$.
- The second inequality implies $m_2^+(\epsilon) \leqslant \epsilon\xi(\rho\epsilon^\eta/3, 0)$. The same argument permits to obtain the existence of $c_1 > 0$ such that $m_2^+(\epsilon) \leqslant c_1\epsilon^{1-\eta}$.

Hence, for ϵ small enough, we deduce the higher-bound $m_2^+(\epsilon) \leqslant \max\{c_1, c_2\}\epsilon^{\Lambda_+(\eta)}$.

Let us now prove the lower-bound. By using the higher-bound, for ϵ small enough, there exists $c_3 > 0$ such that $m_2^+(\epsilon) + \rho\epsilon^\eta/3 \leqslant c_3\epsilon^{\min\{\eta,1/2\}}$. Since ξ is decreasing, we obtain: $m_2^+(\epsilon) \geqslant \epsilon\xi(c_3\epsilon^{\min\{\eta,1/2\}}, \epsilon^{\min\{2\eta,1\}})$. By a classical change of variable, it yields immediately $m_2^+(\epsilon) \geqslant c_4\epsilon^{\Lambda_+(\eta)}$ with $c_4 > 0$. $\square$

The same kind of convergence rate can be analyzed for $m_2^-(\epsilon)$. The proof is quite similar to the one of Proposition 2 so we do not write the details.

Proposition 3. *Let $\eta > 0$ and $\rho > 0$. Then* $\lim_{\epsilon\to 0} \frac{\log m_2^-(\epsilon)}{\log \epsilon} = \Lambda_-(\eta) = \min\{\eta; \frac{1}{2}\}$.

We apply the change of variable $x := \epsilon^{1/4}y$ in (IV) if $\eta \geqslant \frac{1}{2}$. If $\eta \leqslant \frac{1}{2}$, we proceed a *reductio ad absurdum* and we apply the change of variable $x := \epsilon^{\frac{\eta}{2}}y$ in (IV).

Proposition 2 and Proposition 3 point out that the behavior of $m_2^\pm(\epsilon)$ is not symmetric with respect to the critical threshold $\alpha = \theta$. Some heuristic argument which could explain this difference is based on the limit measure. We know that the measure considered when $\alpha \geqslant \theta$ is the trivial measure δ_0 whereas the support of the limit measure for $\alpha < \theta$ contains two points: u_ϵ^- is then farther from δ_0.

We can precise the asymptotic behavior of $m_2^\pm(\epsilon)$. The proof is left to the attention of the reader since the method is the same than the ones used in Proposition 2 and in Proposition 3.

Corollary 4. *Let $\eta > 0$ and $\rho > 0$. Then*

$$\lim_{\epsilon\to 0} \frac{m_2^+(\epsilon)}{\epsilon^{\Lambda_+(\eta)}} = \lambda_+(\eta) := \begin{cases} (2\rho)^{-1} & \text{if } 2\eta < 1 \\ x_\rho & \text{if } 2\eta = 1 \\ x_0 & \text{if } 2\eta > 1 \end{cases} \quad \text{and} \quad \lim_{\epsilon\to 0} \frac{m_2^-(\epsilon)}{\epsilon^{\Lambda_-(\eta)}} = \lambda_-(\eta) := \begin{cases} \rho/5 & \text{if } 2\eta < 1 \\ x_{-\rho} & \text{if } 2\eta = 1 \\ x_0 & \text{if } 2\eta > 1 \end{cases}.$$

Here, for all $r \in \mathbb{R}$, x_r is the unique solution of $x_r = \xi(x_r + r/3, 1)$, see (V) *for the definition of ξ.*

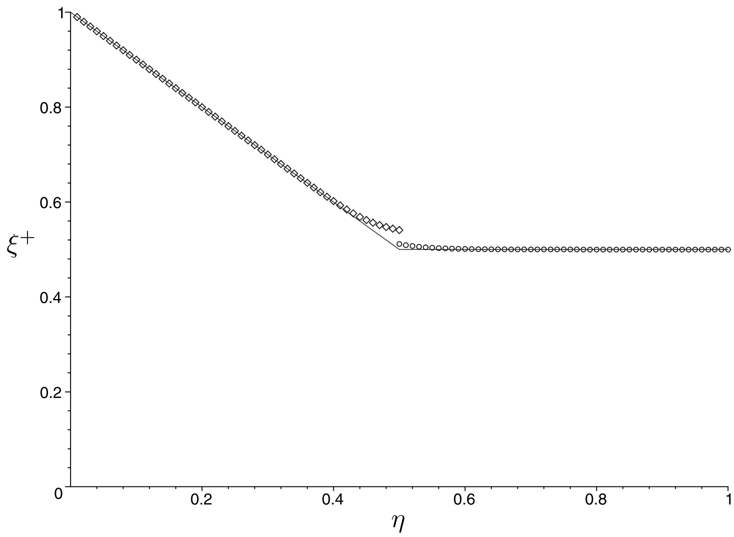

Fig. 2. Description of m_2^+.

By taking $\epsilon := 10^{-10}$, let us simulate $\xi^{\pm}(\eta) := \{\log(m_2^{\pm}(\epsilon)) - \log(\lambda_{\pm})\}/\log\epsilon$ with respect to η (see Figs. 1 and 2). The continuous lines represent $\Lambda_{\pm}(\eta)$. The discontinuity appearing in the simulation is due to the prefactors in the asymptotic estimates.

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