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An inverse source problem with multiple frequency data

*Un problème inverse de source avec des données multi-fréquentielles*Gang Bao^{a,b}, Junshan Lin^b, Faouzi Triki^c^a Department of Mathematics, Zhejiang University, Hangzhou 310027, China^b Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA^c Université Joseph-Fourier, LJK, 38041 Grenoble cedex 9, France

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ABSTRACT

The Note is concerned with an inverse source problem for the Helmholtz equation, which determines the source from measurements of the radiated field away at multiple frequencies. Our main result is a novel stability estimate for the inverse source problem. Our result indicates that the ill-posedness of the inverse problem decreases as the frequency increases. Computationally, a continuation method is introduced to solve the inverse problem by capturing both the macro and the small scales of the source function. A numerical example is presented to demonstrate the efficiency of the method.

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R É S U M É

Dans cette Note on considère un problème inverse de source pour l'équation de Helmholtz. Il consiste à déterminer la fonction source à partir du champ radié loin de la source, et à des multiples fréquences. On donne une nouvelle estimation de stabilité qui montre que la résolution dans la reconstruction de la source s'améliore avec l'augmentation de la fréquence. Ensuite, on propose une méthode de continuation pour résoudre numériquement le problème inverse. Cette méthode permet de capturer à la fois les détails fins et grossiers de la source. Un résultat numérique est présenté afin de montrer l'efficacité de la méthode.

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On considère le problème inverse qui consiste à déterminer la source S d'un champ radié u satisfaisant l'équation de Helmholtz (1). Ce problème est motivé par des applications en imagerie médicale et en conception d'antennes [1–3,7–10].

Soit $\Omega \subset \mathbb{R}^2$ un domaine borné de bord régulier Γ . On suppose que $S(x)$ a un support compact $\tau \subset \Omega \subset \mathbb{R}^2$ vérifiant $\text{dist}(\tau, \Gamma) := \min\{|x - y|; x \in \tau, y \in \Gamma\} > 0$. Il est connu que le système (1) est bien posé. En utilisant la formule de Green, u peut être présenté par (2).

Soit ν le vecteur normal défini sur Γ . La notation u_k est utilisée afin d'exprimer la dépendance de la solution par rapport au nombre d'onde k .

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Le problème inverse multi-fréquentielle de source consiste en la reconstruction de S à partir des mesures de $(u_k, \frac{\partial u_k}{\partial \nu})$ sur le bord Γ pour $k \in [k_{min}, k_{max}]$, avec k_{min} et k_{max} sont deux constantes fixées positives. Plus précisément pour un nombre d'onde donné k on introduit les opérateurs $L_k^{(1)}, L_k^{(2)} : L^2(\tau) \rightarrow L^2(\Gamma)$ définis par (3) et (4).

Le problème inverse peut être réécrit de la façon suivante : trouver $S(x) \in L^2(\tau)$ telles que les équations (5) soient satisfaites simultanément.

Les équations linéaires (5) à une fréquence k fixée possèdent une infinité de solutions. Ceci est du à l'existence de sources non-radiatives, qui produisent des champs identiquement nuls en dehors de τ [6,12]. Le problème est sévèrement mal posé pour des fréquences faibles. En effet, les valeurs singulières de $L_k^{(1)}$ et $L_k^{(2)}$ décroissent exponentiellement lorsque k est proche de zéro (Fig. 1).

On suppose dans la suite que $\tau \subset B_1(0) \subset \Omega$, $k_{min} = 0$ et $k_{max} = c$.

Il est montré dans [4] que le problème inverse multi-fréquence est bien posé. En plus, on a le résultat de stabilité suivante :

Théorème 0.1. Soit S une fonction dans $C^1(\bar{B}_1)$ vérifiant les hypothèses (6) et (7), avec $M > 0$ est une constante fixée.

- A) Si $c \geq \epsilon^{-4} + 1$, alors l'inégalité (8) est valide.
 B) Si $c < \epsilon^{-4} + 1$, alors l'inégalité (9) est valide.

C est une constante universelle.

Pour résoudre le problème inverse, on propose la méthode de continuation. On reconstruit les détails macro de la source à des fréquences basses en utilisant l'approximation de Born [5]. Ensuite, en augmentant la fréquence séquentiellement, on continue à mettre à jour la solution jusqu'à ce que les détails fins de la source soient complètement construits. Le processus est décrit comme suivant : on suppose que pour $k = k_m$, la fonction source est construite avec $S = S_m$. Alors pour une fréquence plus grande $k = k_{m+1} := k_m + \delta k_m$, avec $\delta k_m > 0$ est l'incrément, une itération de type Landweber est appliquée pour résoudre (5) avec $k = k_{m+1}$.

Un exemple numérique est présenté dans les Figs. 2 et 3. Les Fig. 2(a) et Fig. 2(b) sont celles de la source réelle $S(x_1, x_2)$ donnée par l'expression (10). On arrête le processus récursif à la fréquence $k_{max} = 61$. La Fig. 3 représente l'évolution de la construction de la source à différentes valeurs de fréquence. Quand k est relativement petit, on construit uniquement la silhouette de la source. De plus en plus de détails sont construits au fur et à mesure que la fréquence augmente. La construction finale montrée en Fig. 3(f) est très proche de la solution réelle.

1. Formulation

Consider the inverse source problem of determining an unknown scalar source to the homogeneous Helmholtz equation. The study of this inverse problem is motivated particularly by medical imaging and antenna synthesis [1–3,7–10].

Assume that the source function $S(x)$ is compactly supported with support volume $\tau \subset \Omega \subset \mathbb{R}^2$, where Ω is a bounded domain with smooth boundary Γ . It is further assumed that $\text{dist}(\tau, \Gamma) := \min\{|x - y|; x \in \tau, y \in \Gamma\} > 0$. Then the radiating field u generated by the source S satisfies the following Helmholtz equation with the Sommerfeld radiation condition:

$$\begin{cases} \Delta u + k^2 u = S & \text{in } \mathbb{R}^2, \\ \frac{\partial u}{\partial r} - iku = o(r^{-1/2}) & \text{as } r \rightarrow \infty. \end{cases} \quad (1)$$

Here $r = |x|$, k is the wavenumber of the radiating field. If $S \in L^2(\tau)$, the forward problem of determining u for a given source S is well-posed. Moreover, by Green's formula and the radiation condition, the solution to (1) is given by

$$u = \int_{\tau} \Phi(k|x - y|) S(y) dy, \quad (2)$$

where $\Phi(k|x - y|) = -\frac{i}{4} H_0^1(k|x - y|)$ is the fundamental solution for the Helmholtz equation in \mathbb{R}^2 . The function H_0^1 denotes the Hankel function of the first kind of order zero.

Let ν denote the unit outward normal to the boundary Γ . The subscript k is introduced here to denote the radiated field with fixed wavenumber k , i.e. u_k is the solution to (1) with wavenumber k .

The multi-frequency inverse source problem consists of recovering the source function S from the measurement of the radiated wave $(u_k, \frac{\partial u_k}{\partial \nu})$ on the boundary Γ for all $k \in [k_{min}, k_{max}]$, where k_{min} and k_{max} are fixed positive constants. More precisely, for fixed wavenumber k , we define the radiation operators $L_k^{(1)}, L_k^{(2)}$ from $L^2(\tau) \rightarrow L^2(\Gamma)$ as

$$L_k^{(1)}(S)(x) = \int_{\tau} \Phi(k|x - y|) S(y) dy \quad \text{for } x \in \Gamma, \quad (3)$$

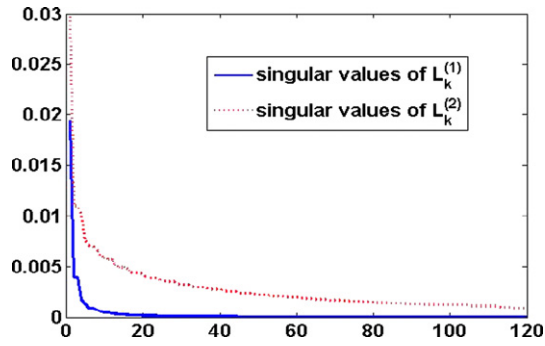


Fig. 1. The first 120 singular values of $L_k^{(1)}$ and $L_k^{(2)}$ when $k = 1$.

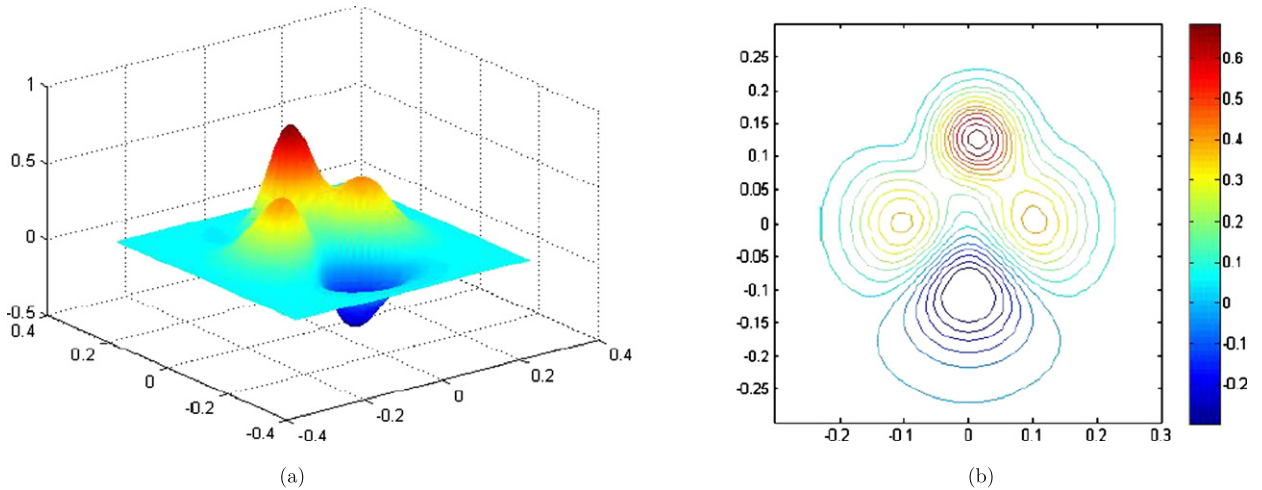


Fig. 2. Real source function (a) and its contour plot (b).

$$L_k^{(2)}(S)(x) = \int_{\tau} \frac{\partial \Phi(k|x-y|)}{\partial \nu(x)} S(y) dy \quad \text{for } x \in \Gamma. \tag{4}$$

Hence, the multi-frequency inverse source problem is to find $S(x) \in L^2(\tau)$ such that the following linear equations are satisfied simultaneously

$$L_k^{(1)}(S)(x) = u_k(x), \quad L_k^{(2)}(S)(x) = \frac{\partial u_k}{\partial \nu}(x) \quad \text{for } x \in \Gamma, \quad k \in [k_{min}, k_{max}]. \tag{5}$$

The solution to linear equations (5) at a single frequency k is not unique, i.e., the source function cannot be uniquely determined from the measurement on the surface Γ . This is due to the existence of the non-radiating source, whose radiating field vanishes identically outside the support volume τ [6,12]. The problem is severely ill-posed for low wavenumbers, that is, an infinitesimal noise in the measurement will give rise to large errors in the reconstruction solution. In fact, it can be shown that the singular values of $L_k^{(1)}$ and $L_k^{(2)}$ decay exponentially for low k (see Fig. 1).

2. Stability

In contrast with the inverse source problem at a fixed frequency, it is proved in [4] that there is a unique source function solution to the multi-frequency inverse source problem.

For simplicity we assume that $\tau \subset B_1(0) \subset \Omega$, $k_{min} = 0$ and $k_{max} = c$. The proof of the following stability estimate is given in [4].

Theorem 2.1. Let S be a function in $C^1(\bar{B}_1)$. Assume that

$$\epsilon = \left\| \frac{\partial u_k}{\partial \nu} \right\|_{L^1(B_c(0) \times \Gamma)} + \|ku_k\|_{L^1(B_c(0) \times \Gamma)} < 1 \tag{6}$$

and

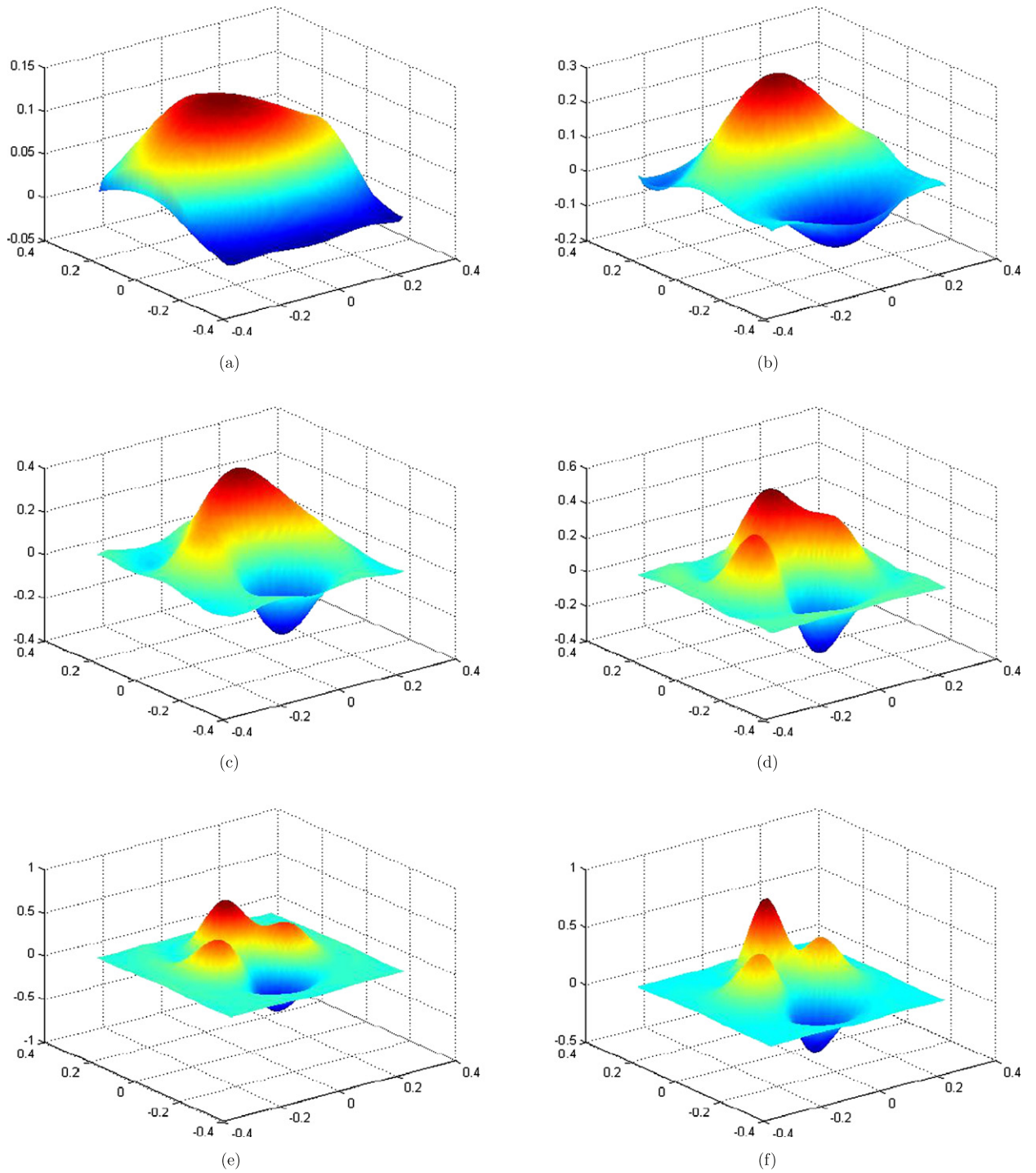


Fig. 3. Reconstructions at (a) $k = 9$, (b) $k = 17$, (c) $k = 25$, (d) $k = 33$, (e) $k = 41$ and (f) $k = 61$.

$$\|S\|_{C^1(\bar{B}_1(0))} \leq M, \tag{7}$$

where M is a positive constant. The following statements hold:

A) If $c \geq \epsilon^{-4} + 1$, then

$$\|S\|_{C^0(\bar{B}_1(0))} \leq C \left(1 + 2M \left(1 + \frac{1}{c} \right) + \left(\frac{1}{16c} \right)^{\frac{3}{2}} \right) \epsilon, \quad (8)$$

where C is a universal constant.

B) If $c < \epsilon^{-4} + 1$, then

$$\|S\|_{C^0(\bar{B}_1(0))} \leq C \left(1 + 2M \left(1 + \frac{1}{c} \right) \left(\frac{1}{4c} + \frac{1}{2\sqrt{c}} \right)^{\frac{1}{6}} \right) \frac{1}{(\ln(\epsilon^{-1}))^{\frac{1}{6}}}, \quad (9)$$

where C is a universal constant.

It follows from the above theorem that the ill-posedness of the inverse problem decreases as the frequency increases. More precisely, we observe that by increasing the frequency, the logarithmic stability converts to a linear one.

3. The continuation method

We next present our reconstruction method, which captures the macro structure of the source at the lower wavenumbers by using the Born approximation [5]. Then, by sequentially increasing the wavenumber, we keep updating the solution until the entire features of the source are successfully reconstructed. The process is described as follows.

Assume that at $k = k_m$, the source function has been recovered with $S = S_m$. Then at a higher wavenumber $k = k_{m+1} := k_m + \delta k_m$, where $\delta k_m > 0$ is the increment, the Landweber iteration is applied to solve (5) with $k = k_{m+1}$. The Landweber iteration can be interpreted as the steepest descent algorithm, which is known to be a regularization scheme for linear ill-posed problems. For the convergence property of the scheme, we refer to [11].

4. A numerical example

In this section, a numerical example is presented to illustrate the efficiency of the continuation method. In the example, $\tau = [-0.3, 0.3] \times [-0.3, 0.3]$, which lies in the domain Ω such that $\text{dist}(\tau, \Gamma) = 0.05$. The measurements $\{u_k, \frac{\partial u_k}{\partial \nu}\}$ are made on Γ for $k \in [k_{\min}, k_{\max}]$. We set $k_{\min} = 1$ in the following numerical example. Assume that the true source

$$S(x_1, x_2) = 1.1e^{-200((x_1-0.01)^2+(x_2-0.12)^2)} - 100(x_2^2 - x_1^2)e^{-90(x_1^2+x_2^2)}. \quad (10)$$

The surface and contour plots of the source are shown in Figs. 2(a), (b). We stop the recursion at $k_{\max} = 61$. Fig. 3 is the evolution of the reconstructed solutions at various wavenumbers. When k is relatively small, only the main profile of the source can be recovered. However, more and more details are captured as k increases, until the reconstructed solution converges to the real one. The final reconstruction image and the corresponding contour plot is shown in Fig. 3(f). It can be seen that the reconstructed solution captures both the macro structures and the small scales of the source.

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