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Combinatorics

Some notes on domination edge critical graphs

Quelques remarques sur les graphes à domination critique par addition d'arête

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ABSTRACT

A graph G is *domination edge critical*, or just γ-edge critical, if for any edge e not in G, $\gamma(G+e) < \gamma(G)$. We will characterize all connected γ-edge critical cactus graphs.

RÉSUMÉ

Un graphe G est un graphe *à domination critique par addition d'arête*, ou simplement γ-critique par arête, si pour toute arête e qui n'est pas dans G on a $\gamma(G+e) < \gamma(G)$. Nous caractérisons les graphes cactus, connexes et γ-critiques par arête.

1. Introduction

For notation and terminology in general we follow [4]. Let $G = (V, E) = (V(G), E(G))$ be a graph without isolated vertices. The (*open*) *neighborhood* $N(v)$ of a vertex $v \in V$ is the set of vertices which are adjacent to v. For a subset S, $N(S) = \bigcup_{v \in S} N(v)$, and $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* of G if $N[S] = V(G)$. The minimum cardinality of a dominating set of G is the *domination number* and denoted by $\gamma(G)$. We refer a dominating set of cardinality $\gamma(G)$ as a $\gamma(G)$-set. For references on domination see for example [4].

The concept of domination critical graphs in sense of edge addition is introduced by Sumner and Blitch [5], and further studied by several authors. A graph G is *domination edge critical*, or just γ-edge critical, if for any edge e not in G, $\gamma(G+e) < \gamma(G)$. It is easy to see that in a γ-edge critical graph G for any edge not in G, $\gamma(G+e) = \gamma(G) - 1$. A graph G is k–γ-edge critical if G is γ-edge critical and $\gamma(G) = k$. For references on domination edge critical graphs see for example [1–3,5,6].

Several authors studied γ-edge critical graphs. Sumner and Blitch [5] characterized γ-edge critical graphs with $\gamma(G) = 1, 2$. They also gave six γ-edge critical graphs of order $n \leqslant 8$ with $\gamma(G) = 3$. For $\gamma(G) \geqslant 3$, characterizing γ-edge critical graphs is difficult and is still open. Sumner [6] characterized disconnected γ-edge critical graphs with $\gamma(G) = 3$. Favaron et al. [3] studied the diameter of γ-edge critical graphs. Ananchuen and Plummer [1,2] studied properties of γ-edge critical graphs with $\gamma(G) = 3$.

A graph G is called a *cactus* graph if each edge of G is contained in at most one cycle. A cactus graph having one cycle is called a *unicyclic* graph and a connected cactus graph with no cycle is called a *tree*. In this paper we study γ-edge critical graphs, and characterize all γ-edge critical cactus graphs.

Let S be a dominating set in a graph G and let $v \in S$. A vertex $w \in V(G)$ is an *S-private neighbor of* v if $N[w] \cap S = \{v\}$. Further, the *S-private neighborhood of* v, denoted $pn[v, S]$, is the set of all S-private neighbors of v. Thus if $pn[v, S] = \{v\}$ then $S - \{v\}$ is a dominating set for $G - v$. We recall that a pendant vertex (or a leaf) is a vertex of degree one, and a

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support vertex is a vertex which is adjacent to a pendant vertex. We also call an edge e a pendant edge if at least one of its end-points is a pendant vertex.

We make use of the following lemma:

Lemma 1. *(See Sumner and Blitch [5].) A graph G with $\gamma(G) = 1$ is γ-edge critical if and only if G is K_n.*

2. Preliminary results

We first give a characterization of all γ-edge critical graphs.

Theorem 2. *A graph G is γ-edge critical if and only if for any two non-adjacent vertices x, y there is a $\gamma(G)$-set S containing x, y such that $pn[x, S] = \{x\}$ or $pn[y, S] = \{y\}$.*

Proof. Let G be a γ-edge critical graph and x, y be two non-adjacent vertices of G. Let S be a $\gamma(G + xy)$-set. If $\{x, y\} \subseteq S$ or $\{x, y\} \cap S = \emptyset$, then S is a dominating set for G, a contradiction. Thus assume, without loss of generality, that $x \in S$ and $y \notin S$. Now $D = S \cup \{y\}$ is a dominating set for G, and $pn[y, D] = \{y\}$. Since $|S| = \gamma(G) - 1$, we obtain that D is a $\gamma(G)$-set.

Conversely, let x, y be two non-adjacent vertices of G. By assumption there is a $\gamma(G)$-set S containing x, y such that $pn[x, S] = \{x\}$ or $pn[y, S] = \{y\}$. Assume that $pn[x, S] = \{x\}$. Then $S - \{x\}$ is a dominating set for $G + xy$. We conclude that G is γ-edge critical. □

In the following observation we characterize γ-edge critical paths and cycles:

Observation 3.

(1) A path P_n is γ-edge critical if and only if $n = 2$.
(2) A cycle C_n is γ-edge critical if and only if $n = 3$ or 4.

To characterizing γ-edge critical trees, we need the following lemmas:

Lemma 4. *Any two support vertices in a γ-edge critical graph are adjacent.*

Proof. Let G be a γ-edge critical and x, y be two support vertices of G. Assume that x is not adjacent to y. By Theorem 2, there is a $\gamma(G)$-set S containing x, y such that $pn[x, S] = \{x\}$ or $pn[y, S] = \{y\}$. Assume that $pn[x, S] = \{x\}$. Then any leaf adjacent to x belongs to S. Now $S - \{w\}$ is a dominating set for G, where w is a leaf adjacent to x which belongs to S. This is a contradiction. □

Similarly the following is verified:

Lemma 5. *Any support vertex in a γ-edge critical graph is adjacent to exactly one leaf.*

We next characterize all γ-edge critical trees.

Theorem 6. *A tree T is γ-edge critical if and only if $T = P_2$.*

Proof. Let T be a γ-edge critical tree. By Lemma 4, $\mathrm{diam}(T) \leqslant 3$. By Lemma 5, T is a path. Now Observation 3 part (1) implies the result. □

3. Main results

In this section we give our main results. We will characterize all connected γ-edge critical cactus graphs. Recall that the corona $\mathrm{cor}(G)$ of a graph G is the graph obtained from G by adding a pendant edge to any vertex of G. We first investigate weather the corona of a graph is γ-edge critical.

Lemma 7. *The corona $\mathrm{cor}(G)$ of a connected graph G is γ-edge critical if and only if G is a complete graph with at least three vertices.*

Proof. Let $\mathrm{cor}(G)$ be γ-edge critical. It is obvious that $\gamma(\mathrm{cor}(G)) = |V(G)|$. If there are two non-adjacent vertices x, y in G, then $\gamma(\mathrm{cor}(G + xy)) = |V(G)| = \gamma(\mathrm{cor}(G))$, a contradiction. Thus G is a complete graph. Assume that $|V(G)| = 2$. Then $\mathrm{cor}(G)$ is the path P_4, and so $\gamma(\mathrm{cor}(G)) = 2$. If x, y are the two end-points of $\mathrm{cor}(G)$, then $\gamma(\mathrm{cor}(G) + xy) = \gamma(C_4) = 2$. This is a contradiction, since G is γ-edge critical. Thus $|V(G)| \geqslant 3$.

Conversely let G be the complete graph with at least three vertices. Let x, y be two leaves of $\mathrm{cor}(G)$ and x_1, y_1 be the support vertices adjacent to x, y, respectively. It follows that $(V(G) - \{x_1, y_1\}) \cup \{x\}$ is a dominating set for $\mathrm{cor}(G) + xy$, and $V(G) - \{y_1\}$ is a dominating set for $\mathrm{cor}(G) + x_1 y$. Since x, y have been chosen arbitrarily, the result follows. □

Lemma 8. *If G is a graph with a path v_1–v_2–v_3–v_4 such that $v_1 \notin N(v_4)$ and $\deg(v_i) = 2$ for $i = 2, 3$, then G is not γ-edge critical.*

Proof. Let G be a graph with a path v_1–v_2–v_3–v_4 such that $\deg(v_i) = 2$ for $i = 2, 3$. Assume that G is γ-edge critical. By Theorem 2, there is a $\gamma(G)$-set S containing v_1, v_4 such that $pn[v_1, S] = \{v_1\}$ or $pn[v_4, S] = \{v_4\}$. Without loss of generality assume that $pn[v_4, S] = \{v_4\}$. Then $S \cap \{v_2, v_3\} \neq \emptyset$. Now $S - \{v_2, v_3\}$ is a dominating set for G, a contradiction. □

Lemma 9. *Let x be a leaf and C be a cycle in a connected graph G such that $d(x, C) \geqslant 2$ and every vertex of C except one is of degree two, then G is not γ-edge critical.*

Proof. Let x be a leaf and C be a cycle in a graph G such that $d(x, C) \geqslant 2$ and every vertex of C except one is of degree two. Assume that G is γ-edge critical. Let $z \in V(C)$ be a vertex with $d(x, z) = d(x, C) = d$, and let P be a shortest path between x and z. Let $b \in N(z)$ be on P. By Lemma 8, $|V(C)| \leqslant 4$. Let $w \in N(z) \cap V(C)$. By Theorem 2, there is a $\gamma(G)$-set S containing w, b such that $pn[w, S] = \{w\}$ or $pn[b, S] = \{b\}$. If $pn[w, S] = \{w\}$, then $|V(C) \cap S| \geqslant 2$, which implies that $|V(C)| = 4$. Now $(S - V(C)) \cup \{v\}$ is a dominating set for G, where $v \in V(C) - N[z]$. This is a contradiction. Thus we may assume that $pn[b, S] = \{b\}$. Now $((S - (V(C) \cup \{b\})) \cup (V(C) - N[z])) \cup \{z\}$ is a dominating set for G. This is a contradiction. □

We are now ready to characterize all unicyclic γ-edge critical graphs.

Theorem 10. *A connected unicycle graph G is γ-edge critical if and only if G is C_3, C_4, or $\mathrm{cor}(C_3)$.*

Proof. First it is easy to see that C_3, C_4, and $\mathrm{cor}(C_3)$ are γ-edge critical. Let G be a unicyclic γ-edge critical graph, and let C be the unique cycle of G. If $G = C$, then by Observation 3 part (2), $G \in \{C_3, C_4\}$. So we assume that $G \neq C$. Let x be a leaf of G such that $d(x, C)$ is maximum, and let $y \in V(C)$ be the vertex with $d(x, y) = d(x, C)$. By Lemmas 4 and 9, $d(x, C) \leqslant 1$ and so $d(x, C) = 1$. By Lemma 5, $\deg(y) = 3$. We show that $G = \mathrm{cor}(C)$. Assume that $G \neq \mathrm{cor}(C)$. Then we assume that C has some vertex of degree 2. By Lemmas 4 and 8, $|V(C)| \leqslant 4$.

If $|V(C)| = 4$, then at most one vertex in $N(y)$ is a support vertex. If there is no support vertex in $N(y)$, then it is easy to see that G is not γ-edge critical. We may now assume that there is a support vertex $a \in N(y) \cap V(C)$. Let a_1 be the leaf adjacent to a. Then $\gamma(G) = \gamma(G + a_1 x) = 2$, a contradiction.

Thus we assume that $|V(C)| = 3$. Let $V(C) = \{y, a, b\}$. Since G has some vertex of degree 2, we assume that $\deg(b) = 2$. If $\deg(a) = 2$, then $\gamma(G) = 1$ and by Lemma 1, G is not γ-edge critical. So assume that $\deg(a) = 3$. Let a_1 be the leaf adjacent to a. Then $\gamma(G) = \gamma(G + a_1 x) = 2$, a contradiction.

We conclude that $G = \mathrm{cor}(C)$. Then by Lemma 7, $|V(C)| = 3$ and so $G = \mathrm{cor}(C_3)$. □

Our next aim is to characterize all γ-edge critical cactus graphs with at least two cycles.

Lemma 11. *If G is a γ-edge critical cactus graph with at least two cycles, then $\delta(G) \geqslant 2$.*

Proof. Let G be a γ-edge critical cactus graph with $k \geqslant 2$ cycles. Let $C_1, C_2, \ldots, C_k$ be the cycles of G. Assume that $\delta(G) = 1$. Let x be a leaf of G. Without loss of generality assume that $d(x, C_1) \leqslant d(x, C_i) \leqslant d(x, C_2)$ for $i = 1, 2, \ldots, k$. Let $z \in V(C_2)$ be the vertex with $d(x, z) = d(x, C_2) = d$, and let P be the shortest path between x and z. If $d(x, z) \geqslant 2$, then by Lemma 4, any vertex of $V(C_2) - \{z\}$ is of degree two, and by Lemma 9, G is not γ-edge critical which is a contradiction. Thus $d(x, z) \leqslant 1$.

Suppose next that $d(x, z) = 1$. Thus $d(x, C_i) = 1$ for $i = 1, 2, \ldots, k$, and $V(C_1) \cap V(C_2) \cap \cdots \cap V(C_k) = \{z\}$. By Lemma 5, x is the only leaf adjacent to z. Let $w_1 \in N(z) \cap V(C_1)$ and $w_2 \in N(z) \cap V(C_2)$. By Theorem 2, there is a $\gamma(G)$-set S containing w_1, w_2 such that $pn[w_1, S] = \{w_1\}$ or $pn[w_2, S] = \{w_2\}$. But then $(S - \{w_1, w_2, x\}) \cup \{z\}$ is a dominating set for G, a contradiction. We deduce that $d = 0$, contradicting that x is a leaf. □

Theorem 12. *There is no γ-edge critical cactus graph with at least two cycles.*

Proof. Assume to the contrary that G is a γ-edge critical cactus graph with at least two cycles. Let $C_1, C_2, \ldots, C_k$ be the cycles of G. By Lemma 11, $\delta(G) \geqslant 2$. Without loss of generality assume that $d(C_1, C_2) \leqslant d(C_i, C_j)$ for $1 \leqslant i, j \leqslant k$ and $i \neq j$. By Lemma 8, $|V(C_i)| \leqslant 4$ for $i = 1, 2$. Let $x \in V(C_1)$ and $y \in V(C_2)$ be two vertices with $d(x, y) = d(C_1, C_2)$.

We show that $d(x, y) = 0$. Assume that $d(x, y) \geqslant 1$. Let $a \in N(x) \cap V(C_1)$ and $b \in N(x) \cap V(P)$. By Theorem 2, there is a $\gamma(G)$-set S containing a, b such that $pn[a, S] = \{a\}$ or $pn[b, S] = \{b\}$. Suppose that $pn[b, S] = \{b\}$. Then $(S - (V(C_1) \cup \{b\})) \cup \{x\} \cup (V(C_1) - N[x])$ is a dominating set for G, a contradiction. Thus $pn[a, S] = \{a\}$. Then $|S \cap V(C_1)| \geqslant 2$. This implies that $|V(C_1)| = 4$. Now $(S - V(C_2) \cup (V(C_2) - N[x]))$ is a dominating set for G, a contradiction. Hence $d(x, y) = 0$.

This implies that $V(C_1) \cap V(C_2) \cap \cdots \cap V(C_k) = \{x\}$. Let $a_1 \in N(x) \cap V(C_1)$ and $b_1 \in N(x) \cap V(C_2)$. By Theorem 2, there is a $\gamma(G)$-set S containing a_1, b_1 such that $pn[a_1, S] = \{a_1\}$ or $pn[b_1, S] = \{b_1\}$. But $|V(C_i)| \leqslant 4$ for $i = 1, 2$. Now it is a routine matter to see that G is not γ-edge critical, a contradiction. □

Now from Theorems 6, 10, and 12 we obtain the following:

Theorem 13. *A connected cactus graph G is γ-edge critical if and only if G is P_2, C_3, C_4, or* $\mathrm{cor}(C_3)$.

We close with the following problem:

Problem 14. Characterize all connected γ-edge critical graphs G with $\delta(G) = 1$.

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