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The absolute stable rank of $C(X, \tau)$ *Le rang stable absolu de $C(X, \tau)$*

Raymond Mortini, Jérôme Noël

Département de mathématiques, LMAM, UMR 7122, université Paul-Verlaine, Ile du Saulcy, 57045 Metz, France

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ABSTRACT

Let X be a compact Hausdorff space and τ a topological involution on X . Let $C(X, \tau)$ be the real algebra of all complex-valued continuous functions on X that satisfy $f(\tau(x)) = \overline{f(x)}$ for every $x \in X$. It is shown that the absolute stable rank of $C(X, \tau)$ equals the Bass, and hence topological stable rank of $C(X, \tau)$.

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R É S U M É

Soit X un espace de Hausdorff et τ une involution topologique sur X . Soit $C(X, \tau)$ l'algèbre réelle de toutes les fonctions continues à valeurs complexes sur X telles que $f(\tau(x)) = \overline{f(x)}$ pour tout $x \in X$. Dans un papier récent, le premier auteur de cette Note et R. Rupp ont pu calculer les rangs stables de Bass et topologique de $C(X, \tau)$. Nous montrons ici que le rang stable absolu de $C(X, \tau)$ coïncide avec le rang stable de Bass, et ainsi aussi avec le rang stable topologique de $C(X, \tau)$. On profite de cette Note pour annoncer ainsi ce théorème de Mortini–Rupp qui va apparaître ailleurs.

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1. Introduction

The concept of the stable rank of a ring, introduced by H. Bass in [1], has proved to be very useful in algebraic K -theory. Several notions of stable rank have been developed meanwhile; among them the topological [6] and absolute stable rank [3,7]. In this Note, we are interested in these notions for function spaces. Our single goal will be to determine the absolute stable rank for the real function-algebra $C(X, \tau)$, that is defined by $C(X, \tau) = \{f \in C(X, \mathbb{C}) : f \circ \tau = \overline{f}\}$, where X is a compact Hausdorff space and τ is a topological involution on X .

This algebra plays a fundamental role in the theory of real function algebras (see the monograph [2] by Kulkarni and Limaye) and actually is a more general object than $C(X, \mathbb{C})$ and $C(X, \mathbb{R})$. In a recent paper [5], it was shown that

$$\text{bsr } C(X, \tau) = \text{tsr } C(X, \tau) = \max \left\{ \left\lceil \frac{\dim X}{2} \right\rceil, \dim E \right\} + 1,$$

where E denotes the set of fixed points of τ and $\dim X$ the covering dimension of X . In this Note, we are going to show that $\text{asr } C(X, \tau) = \text{bsr } C(X, \tau) = \text{tsr } C(X, \tau)$.

As a by-product, our Note will provide at the same time shorter proofs of the corresponding results by Vaserstein et al. on the absolute stable ranks of $C(X, \mathbb{C})$ and $C(X, \mathbb{R})$ (see [7] and [8]).

E-mail addresses: mortini@univ-metz.fr (R. Mortini), jerome.noel@univ-metz.fr (J. Noël).

2. Notations and definitions

Let A be a commutative real or complex Banach algebra. An n -tuple $(a_1, \dots, a_n) \in A^n$ is said to be *invertible* if there exists $(b_1, \dots, b_n) \in A^n$ such that $\sum_{j=1}^n a_j b_j = 1$, where 1 denotes the unit element in A . The set of all invertible n -tuples is denoted by $U_n(A)$. An invertible $(n + 1)$ -tuple (a_1, \dots, a_{n+1}) is said to be *reducible*, if there exists $(b_1, \dots, b_n) \in A^n$ such that $(a_1 + b_1 a_{n+1}, \dots, a_n + b_n a_{n+1}) \in U_n(A)$.

The smallest integer n for which every element in $U_{n+1}(A)$ is reducible, is called the *Bass stable rank* of A , and is denoted by $\text{bsr } A$. If no such n exists, then $\text{bsr } A = \infty$.

The following notion has been introduced by M. Rieffel [6]. The *topological stable rank* of A , denoted by $\text{tsr } A$, is the smallest integer n for which $U_n(A)$ is dense in A^n . If no such n exists, then $\text{tsr } A = \infty$. Moreover, it is well known that $\text{bsr } A \leq \text{tsr } A$.

Finally, we define the notion of absolute stable rank. We give a version that is more transparent than the original definition given in [3] and [7]. Later on, we will observe that both versions are equivalent, though.

Let $M(A)$ be the space of (nonzero) multiplicative \mathbb{C} -linear, respectively \mathbb{R} -linear, functionals with target space \mathbb{C} . Note that in the real-Banach algebra setting, \mathbb{C} is considered as a 2-dimensional vector space over \mathbb{R} .

Let $\hat{f} : M(A) \rightarrow \mathbb{C}, m \mapsto m(f)$ denote the Gelfand transform of $f \in A$, and let $Z(f) = \{m \in M(A) : \hat{f}(m) = 0\}$. The *absolute stable rank*, $\text{asr } A$, of A now is the smallest integer n such that for all $(a_1, \dots, a_{n+1}) \in A^{n+1}$, there exists $(b_1, \dots, b_n) \in A^n$ such that $\sum_{j=1}^n |\hat{a}_j + \hat{b}_j \hat{a}_{n+1}| > 0$ outside $Z(a_{n+1})$.

For the reader's convenience, we present here a proof that the notion of absolute stable rank defined above is equivalent to that appearing in [7] and [3].

Observation 2.1. *Let A be a commutative unital (real or complex) Banach algebra. The following assertions are equivalent:*

- (1) $\forall (a_1, \dots, a_n, a_{n+1}) \in A^{n+1} \exists (x_1, \dots, x_n) \in A^n \forall h \in A : \sum_{j=1}^n (a_j + x_j a_{n+1})A + (1 + h a_{n+1})A = A$;
- (2) $\forall (a_1, \dots, a_n, a_{n+1}) \in A^{n+1} \exists (x_1, \dots, x_n) \in A^n : \sum_{j=1}^n |\hat{a}_j + \hat{x}_j \hat{a}_{n+1}| > 0$ on $M(A) \setminus Z(a_{n+1})$.

Proof. (2) \Rightarrow (1) Obviously our assumption (2) implies that for all $h \in A$ $|1 + \hat{h} \hat{a}_{n+1}| + \sum_{j=1}^n |\hat{a}_j + \hat{x}_j \hat{a}_{n+1}| > 0$ on $M(A)$. Hence (1) holds.

$\neg(2) \Rightarrow \neg(1)$ Let $m \in M(A) \setminus Z(a_{n+1})$ satisfy $(\hat{a}_j + \hat{x}_j \hat{a}_{n+1})(m) = 0$ for all $j = 1, \dots, n$. Choose $h \in A$ so that $\hat{h}(m) = -\hat{a}_{n+1}(m)^{-1}$.¹ Then $(1 + \hat{h} \hat{a}_{n+1})(m) = 0$. Hence $1 + h a_{n+1}$ and all of the $a_j + x_j a_{n+1}$ belong to a common maximal ideal. Thus $\neg(1)$ holds. \square

It is easily seen from the observation above (and well known) that $\text{bsr } A \leq \text{asr } A$.

Is there a relation between $\text{asr } A$ and $\text{tsr } A$?

3. Invertible extensions

It is well known that stable rank questions are intimately related to questions of extensions of invertible tuples. Our main tool, Lemma 3.1, will show the existence of invertible extensions F that not only remain close to any fixed Tietze extension of a given invertible tuple on some closed subset M of X , but whose lower bound $\inf_X |F|$ is also controllable.

Recall that a subset M of X is τ -invariant, if $\tau(M) = M$. For a τ -invariant set $V \subseteq X$, let $C_b(V, \tau)$ be the space of all bounded continuous functions on V that satisfy $f \circ \tau = \bar{f}$.

For $\mathbf{f} = (f_1, \dots, f_m)$ let $|\mathbf{f}| = \sqrt{\sum_{j=1}^m |f_j|^2}$. Finally, the set of interior points of a subset S of the topological space X is denoted by S° .

Lemma 3.1. *Let X be a compact Hausdorff space, $0 < \varepsilon \leq 1$, and let M be a closed τ -invariant subset of X . Suppose that $m = \text{bsr } C(X, \tau) < \infty$. Let \mathbf{f} be an m -tuple in $C(X, \tau)$ such that $|\mathbf{f}| > \varepsilon > 0$ on M . Then $\mathbf{f}|_M$ admits an invertible extension $\mathbf{F} \in U_m(C(X, \tau))$ such that $|\mathbf{f} - \mathbf{F}| < 6\varepsilon$ and $|\mathbf{F}| > \varepsilon$.*

Proof. Since M is compact, we may choose ε_j so that $0 < \varepsilon < \varepsilon_1 < \varepsilon_2 < 2\varepsilon$ and $|\mathbf{f}| > \varepsilon_2$ on M . Let $E_{\varepsilon_1} = \{x \in X : |\mathbf{f}(x)| \geq \varepsilon_1\}$. Hence, by the choice of ε_2 , $M \subseteq E_{\varepsilon_1}$. We claim that there exists an extension $\hat{\mathbf{f}} \in C(X, \tau)^m$ of $\phi := \mathbf{f}/|\mathbf{f}| \in C(E_{\varepsilon_1}, \tau)^m$ such that $|\hat{\mathbf{f}}| = 1$.

Indeed, let $\Phi = (\Phi_1, \dots, \Phi_m)$ be a τ -invariant Tietze extension of ϕ to X and let $G \in C(X, \tau)$ be chosen so that $0 \leq G \leq 1$, $G \equiv 0$ on E_{ε_1} and $G \equiv 1$ on $\bigcap_{j=1}^m Z(\Phi_j)$. Then the $(m + 1)$ -tuple (Φ, G) is invertible in $C(X, \tau)$. Since $\text{bsr } C(X, \tau) = m$, there

¹ This can be seen as follows: since $w := \hat{a}_{n+1}(m)$ and $w + 1$ are linearly independent in \mathbb{C} over \mathbb{R} whenever w is non-real, every $c \in \mathbb{C}$ admits the representation $c = \alpha w + \beta(w + 1)$, $\alpha, \beta \in \mathbb{R}$. Thus $h = \alpha a_{n+1} + \beta(a_{n+1} + 1)$ does the job.

is an m -tuple \mathbf{H} in $C(X, \tau)$ such that $\Phi + \mathbf{G}\mathbf{H}$ is invertible in $C(X, \tau)$. Now the m -tuple $\hat{\mathbf{f}} := (\Phi + \mathbf{G}\mathbf{H})/|\Phi + \mathbf{G}\mathbf{H}|$ is the desired extension.

Next, let $K \in C(X, \tau)$ be chosen so that $0 \leq K \leq 1$, $K \equiv 0$ on M and $K \equiv 1$ on $|\mathbf{f}| \leq \varepsilon_1$. Then, as we are going to show, the m -tuple $\mathbf{F} := (|\mathbf{f}| + \varepsilon_1 K)\hat{\mathbf{f}}$ is the desired approximation of \mathbf{f} .

In fact, on M , $\mathbf{F} = |\mathbf{f}|\hat{\mathbf{f}} = |\mathbf{f}|(\Phi/|\Phi|) = \mathbf{f}$.

Moreover, since $|\hat{\mathbf{f}}| = 1$ and $K \geq 0$, we have $|\mathbf{F}(x)| \geq \varepsilon_1$ whenever $|\mathbf{f}(x)| \geq \varepsilon_1$ and $|\mathbf{F}(x)| \geq \varepsilon_1 K(x) = \varepsilon_1 > \varepsilon$ whenever $|\mathbf{f}(x)| \leq \varepsilon_1$. Thus $|\mathbf{F}| > \varepsilon$.

Finally, we shall see that $|\mathbf{F} - \mathbf{f}| < 6\varepsilon$.

Indeed, on E_{ε_1} , $\hat{\mathbf{f}} = \mathbf{f}/|\mathbf{f}|$, and so, $|\mathbf{F} - \mathbf{f}| = ||\mathbf{f}|\hat{\mathbf{f}} + \varepsilon_1 K\hat{\mathbf{f}} - \mathbf{f}| = \varepsilon_1 K|\hat{\mathbf{f}}| \leq \varepsilon_1 \leq 2\varepsilon$.

On the other hand, if $0 \neq |\mathbf{f}(x)| \leq \varepsilon_1$, then $|\mathbf{f} - \mathbf{F}| \leq |\mathbf{f}|\left|\frac{\mathbf{f}}{|\mathbf{f}|} - \hat{\mathbf{f}}\right| + \varepsilon_1 K|\hat{\mathbf{f}}| \leq \varepsilon_1 \cdot 2 + \varepsilon_1 = 3\varepsilon_1 < 6\varepsilon$, and if $|\mathbf{f}(x)| = 0$, then the assertion is obvious, since $|\hat{\mathbf{f}}| = 1$. \square

As a corollary we obtain the following extension of Lemma 3.1:

Corollary 3.2. *Let Y be a compact Hausdorff space for which $m := \text{bsr } C(Y, \tau) < \infty$. Let $g \in C(Y, \tau)$ and $V = Y \setminus Z(g)$. Suppose that M, R, S are compact τ -invariant subsets of V satisfying $\emptyset \neq M \subseteq R \subseteq S^\circ \subseteq S \subseteq V$. Let $\mathbf{b} \in C(V, \tau)^m$ and assume that for some $\varepsilon > 0$, $|\mathbf{b}| > \varepsilon$ on M . Then there is $\mathbf{B} \in C(V, \tau)^m$ such that*

- (1) $|\mathbf{B} - \mathbf{b}| < 6\varepsilon$;
- (2) $|\mathbf{B}| \geq \varepsilon$ on R ;
- (3) $\mathbf{B} = \mathbf{b}$ on $M \cup (V \setminus S)$.

Proof. Let $\alpha \in C(Y, \tau)$ satisfy $\alpha \equiv 0$ on $V \setminus S$ and $\alpha \equiv 1$ on R . Let $\mathbf{F} \in C(S, \tau)^m$ be the function of Lemma 3.1 (where we have put $X = S$ and $\mathbf{f} = \mathbf{b}$). Then $\mathbf{B} = \alpha\mathbf{F} + (1 - \alpha)\mathbf{b}$ does the job. \square

4. The main result

As in [5], let E denote the set of fixed points of the topological involution τ on X .

Theorem 4.1. $\text{bsr } C(X, \tau) = \text{tsr } C(X, \tau) = \text{asr } C(X, \tau) = \max\{\lfloor \frac{\dim X}{2} \rfloor, \dim E\} + 1$.

Proof. Suppose that $m = \text{bsr } C(X, \tau) < \infty$. Let $(\mathbf{a}, g) = (a_1, \dots, a_m, g)$ be an $(m + 1)$ -tuple in $C(X, \tau)$. Let $V = X \setminus Z(g)$. We have to prove the existence of $h_j \in C(X, \tau)$ such that $\sum_{j=1}^m |a_j + gh_j| > 0$ on V .

To this end, let $\mathbf{f} = \mathbf{a}/g^2$ and $\varepsilon \in]0, 1]$. Then $\mathbf{f} \in C(V, \tau)^m$. If $\mathbf{a} = \mathbf{0}$, then we let $h_j = 1$. If $\mathbf{f}(x_0) \neq \mathbf{0}$, we choose ε so that $0 < \varepsilon < |\mathbf{f}(x_0)|$.

Let $V = \bigcup_{n=0}^\infty X_n$, where $X_0 = \{x_0, \tau(x_0)\}$ and where each X_n is τ -invariant and compact. We may assume that $X_n \subseteq X_{n+1}^\circ$ for all n . Let $\mathbf{F}_0 = \mathbf{f}$. By Corollary 3.2, applied to $M = X_{n-1}$, $R = X_n$ and $S = X_{n+1}$, there is a sequence \mathbf{F}_n of m -tuples in $C(V, \tau)$ such that $|\mathbf{F}_n - \mathbf{F}_{n-1}| < 6\varepsilon$; $\min_{X_n} |\mathbf{F}_n| > \varepsilon$ and $\mathbf{F}_n = \mathbf{F}_{n-1}$ on $X_{n-1} \cup (V \setminus X_{n+1})$.

Now let $\mathbf{F}(x) = \mathbf{f}(x)$ if $x \in X_0$ and $\mathbf{F}(x) = \mathbf{F}_n(x)$ if $x \in X_n \setminus X_{n-1}$ ($n = 1, 2, \dots$). Then \mathbf{F} is well defined on V and hence $\mathbf{F} \in C(V, \tau)^m$. Moreover, $|\mathbf{F} - \mathbf{f}| \leq 12\varepsilon$ and $|\mathbf{F}| \geq \varepsilon$.

Hence $|g^2\mathbf{F} - \mathbf{a}| = |g^2\mathbf{F} - g^2\mathbf{f}| < 12|g|^2$ and $|g^2\mathbf{F}| > 0$ on V . Moreover, $|g^2\mathbf{F}| \leq |\mathbf{a}| + 12\|g\|_X^2$; so $g^2\mathbf{F} \in C_b(V, \tau)^m$.

Also, $\mathbf{K} := \frac{g^2\mathbf{F} - \mathbf{a}}{g^2}$ is continuous on V and bounded by 12. Hence $\mathbf{h} := g\mathbf{K} \in C(X, \tau)^m$.

Thus $g^2\mathbf{F} = \mathbf{a} + g^2\mathbf{K} = \mathbf{a} + g(g\mathbf{K}) = \mathbf{a} + g\mathbf{h}$. We conclude that $|\mathbf{a} + g\mathbf{h}| > 0$ on V .

To sum up, we have shown that $\text{asr } C(X, \tau) \leq m = \text{bsr } C(X, \tau)$. The reverse inequality always being true, we get the assertion from [5]. \square

5. An alternative proof

In this section we present an alternative proof, based on the Swan–Vaserstein method, of Theorem 4.1. First we derive from Lemma 3.1 the following result on the topological stable rank of the algebra $C_b(V, \tau)$, where V is some open subset of X :

Corollary 5.1. *Let V be an open F_σ -subset of X . Then $\text{tsr } C_b(V, \tau) \leq \text{tsr } C(X, \tau)$.*

² Note that for $x \in X_n \setminus X_{n-1}$ we have $|\mathbf{F}_n(x) - \mathbf{f}(x)| \leq |\mathbf{F}_n(x) - \mathbf{F}_{n-1}(x)| + |\mathbf{F}_{n-1}(x) - \mathbf{F}_{n-2}(x)| + \underbrace{|\mathbf{F}_{n-2}(x) - \mathbf{f}(x)|}_{=0} \leq 6\varepsilon + 6\varepsilon = 12\varepsilon$.

Proof. Let $m := \text{tsr } C(X, \tau) < \infty$. Let $\mathbf{f} = (f_1, \dots, f_m)$ be an m -tuple in $C_b(V, \tau)$. Fix $\varepsilon \in]0, 1]$.

If $\mathbf{f} = \mathbf{0}$, then we let $\mathbf{F} = (\varepsilon, 0, \dots, 0)$. If $\mathbf{f} \neq \mathbf{0}$, let x_0 satisfy $|\mathbf{f}(x_0)| \neq 0$. By passing to a smaller epsilon, if necessary, we may assume that $|\mathbf{f}(x_0)| > \varepsilon/12$. As above, there is $\mathbf{F} \in C_b(V, \tau)^m$ so that on V , $|\mathbf{F} - \mathbf{f}| \leq \varepsilon$ and $|\mathbf{F}| \geq \varepsilon/12$. In particular, \mathbf{F} is invertible in $C_b(V, \tau)$. Thus $\text{tsr } C_b(V, \tau) \leq m = \text{tsr } C(X, \tau)$. \square

Second proof of Theorem 4.1. Let $m = \text{bsr } C(X, \tau) = \text{tsr } C(X, \tau)$ and let $\mathbf{f} = (f_1, \dots, f_m, f_{m+1})$ be an $(m + 1)$ -tuple in $C(X, \tau)$. We have to prove the existence of $h_j \in C(X, \tau)$ such that $\sum_{j=1}^m |f_j + h_j f_{m+1}| > 0$ outside $Z(f_{m+1})$.

For this, let $\mathbf{F} = (f_1, \dots, f_m, |\mathbf{f}|f_{m+1})$. Then $|\mathbf{F}|$ is clearly in $C(X, \tau)$. Let $V = \{x \in X, \mathbf{f}(x) \neq \mathbf{0}\}$. For later purpose we note that $\{x \in X: f_{m+1}(x) \neq 0\} \subseteq V$.

Since \mathbf{f} has the same zeros as \mathbf{F} , we also have $V = \{x \in X, \mathbf{F}(x) \neq \mathbf{0}\}$. Let the $(m + 1)$ -tuple $\mathbf{a} = (a_1, \dots, a_{m+1})$ be defined on V by

$$a_j = \begin{cases} \frac{f_j}{|\mathbf{F}|} & \text{if } j \in \{1, \dots, m\}, \\ \frac{f_{m+1}|\mathbf{f}|}{|\mathbf{F}|} & \text{if } j = m + 1. \end{cases}$$

Then, by definition, $\mathbf{a} \in C_b(V, \tau)^{m+1}$. We note that the factor $|\mathbf{f}|$ is needed later to multiply a bounded continuous function on V to a continuous function on $V \cup Z(|\mathbf{f}|) = X$. Also, the division by $|\mathbf{F}|$ is needed to get an invertible $(m + 1)$ -tuple on V . In fact, because $\sum_{j=1}^{m+1} |a_j|^2 = 1$, the tuple \mathbf{a} is unimodular. Thus \mathbf{a} is an invertible $(m + 1)$ -tuple; that is $\mathbf{a} \in U_{m+1}(C_b(V, \tau))$.

Now, by Corollary 5.1, $\text{bsr } C_b(V, \tau) \leq \text{tsr } C_b(V, \tau) \leq \text{tsr } C(X, \tau) = \text{bsr } C(X, \tau) = m$. Hence there exists $\mathbf{b} = (b_1, \dots, b_m) \in C_b(V, \tau)^m$ such that $(a_1 + b_1 a_{m+1}, \dots, a_m + b_m a_{m+1}) \in U_m(C_b(V, \tau))$. Let $\mathbf{h} = (h_1, \dots, h_m)$ be the m -tuple defined on X by

$$h_j = \begin{cases} |\mathbf{f}|b_j & \text{on } V, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, \mathbf{h} is in $C(X, \tau)^m$. Hence $f_j + h_j f_{m+1} \in C(X, \tau)$ and $(f_j + h_j f_{m+1})|_V = (a_j + b_j a_{m+1})|\mathbf{F}|$.

Note that the m -tuple $\mathbf{G} := (a_1 + b_1 a_{m+1}, \dots, a_m + b_m a_{m+1})|\mathbf{F}| \in U_m(C(V, \tau))$.

Since $\{x \in X: f_{m+1}(x) \neq 0\} \subseteq V$, we therefore have found $\mathbf{h} = (h_1, \dots, h_m) \in C(X, \tau)^m$ such that $|\mathbf{f} + f_{m+1}\mathbf{h}| > 0$ outside $Z(f_{m+1})$. Thus $\text{asr } C(X, \tau) \leq m$. \square

6. The algebra $C(K)_{\text{sym}}$

In this part, we are going to mention a particular case of $C(X, \tau)$. Let $K \subseteq \mathbb{C}$ be a *real-symmetric* compact set, that is to say that $\bar{z} \in K$ whenever $z \in K$ and τ is given by $\tau(z) = \bar{z}$. We obtain the algebra $C(K, \tau)$ denoted by

$$C(K)_{\text{sym}} = \{f \in C(K, \mathbb{C}): f(z) = \overline{f(\bar{z})}\}.$$

The Bass and topological stable ranks for this algebra have recently been determined (see [4]). In view of our main Theorem 4.1 above, we may also deduce the absolute stable rank of $C(K)_{\text{sym}}$.

Corollary 6.1. *The absolute stable rank of $C(K)_{\text{sym}}$ is given by:*

- (1) $\text{asr } C(K)_{\text{sym}} = \text{bsr } C(K)_{\text{sym}} = \text{tsr } C(K)_{\text{sym}} = 1$ if and only if the interior $\mathring{K} = \emptyset$ and $K \cap \mathbb{R}$ is totally disconnected or empty;
- (2) $\text{asr } C(K)_{\text{sym}} = \text{bsr } C(K)_{\text{sym}} = \text{tsr } C(K)_{\text{sym}} = 2$ if and only if the interior $\mathring{K} \neq \emptyset$ or $K \cap \mathbb{R}$ contains an interval.

Proof. The set of fixed points of τ is $K \cap \mathbb{R}$ or empty. Since $\dim(K \cap \mathbb{R}) \leq \dim \mathbb{R} = 1$, we get

$$\dim(K \cap \mathbb{R}) = \begin{cases} 0 & \text{if } K \cap \mathbb{R} \text{ is totally disconnected or empty,} \\ 1 & \text{if } K \cap \mathbb{R} \text{ contains an interval.} \end{cases}$$

Moreover, $\dim K = 2$ if and only if $\mathring{K} \neq \emptyset$. Hence it suffices to apply Theorem 4.1 to have the result. \square

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