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Probability Theory

# Tail behavior of laws stable by random weighted mean

## *Variation régulière des lois stables par moyenne pondérée aléatoire*

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ABSTRACT

Let $(N, A_1, A_2, \ldots)$ be a sequence of random variables with $N \in \mathbb{N} \cup \{\infty\}$ and $A_i \in \mathbb{R}_+$. We are interested in asymptotic properties of solutions of the distributional equation $Z = \sum_{i=1}^{N} A_i Z_i$, where $Z_i$ are nonnegative random variables independent of each other and independent of $(N, A_1, A_2, \ldots)$, each has the same distribution as $Z$ which is unknown. For a solution $Z \geqslant 0$ with finite mean, we show that under a natural moment condition, the regular variation of $\mathbb{P}(Z > x)$ $(x \to \infty)$ is equivalent to that of $\mathbb{P}(Y_1 > x)$, where $Y_1 = \sum_{i=1}^{N} A_i$. The results generalize the corresponding theorems of Bingham and Doney (1974, 1975) [1,2] and de Meyer (1982) [6] on Galton–Watson processes and Crump–Mode–Jirina processes, and improve those of Iksanov and Polotskiy (2006) [7] on branching random walks.



RÉSUMÉ

Soit $(N, A_1, A_2, \ldots)$ une suite de variables aléatoires avec $N \in \mathbb{N} \cup \{\infty\}$ et $A_i \in \mathbb{R}_+$. Nous nous sommes intéressés aux propriétés asymptotiques des solutions de l'équation en distribution $Z = \sum_{i=1}^{N} A_i Z_i$, où $Z_i$ sont des variables aléatoires non-négatives, mutuellement indépendantes et indépendantes de $(N, A_1, A_2, \ldots)$, chacune a la même loi que $Z$ qui est inconnue. Pour une solution $Z \geqslant 0$ de moyenne finie, nous montrons que sous une condition de moment naturelle, la variation régulière de la probabilité de queue $\mathbb{P}(Z > x)$ $(x \to \infty)$ est équivalente à celle de $\mathbb{P}(Y_1 > x)$, où $Y_1 = \sum_{i=1}^{N} A_i$. Les résultats généralisent les théorèmes correspondants de Bingham et Doney (1974, 1975) [1,2] et de Meyer (1982) [6] sur les processus de Galton–Watson et de Crump–Mode–Jirina, et améliorent ceux d'Iksanov et Polotskiy (2006) [7] sur les marches aléatoires branchantes.



### Version française abrégée

Soit $(N, A_1, A_2, \ldots)$ une suite de variables aléatoires avec $N \in \mathbb{N} \cup \{\infty\}$ et $A_i \in \mathbb{R}_+$, où $\mathbb{N} = \{0, 1, 2, \ldots\}$ et $\mathbb{R}_+ = [0, \infty)$. Supposons que $\mathbb{E} \sum_{i=1}^{N} A_i = 1$, et considérons l'équation en distribution

$$Z = \sum_{i=1}^{N} A_i Z_i, \tag{0.1}$$

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où $Z_i$ sont des variables aléatoires non-négatives, mutuellement indépendantes et inépendantes de la suite $(N, A_1, A_2, \ldots)$, chacune a la même loi que $Z$ qui est inconnue. En termes de la transformée de Laplace $\phi(s) = \mathbb{E}e^{-sZ}$, l'équation (0.1) s'écrit

$$\phi(s) = \mathbb{E}\prod_{i=1}^{N}\phi(sA_i), \quad s \geqslant 0. \tag{0.2}$$

L'étude de l'équation (0.1) est intéressante grâce aux applications dans des modèles divers, comprenant en particulier des processus de branchement, des cascades multiplicatives et des marches aléatoires branchantes. Ici nous étudions la variation régulière de la probabilité de queue $\mathbb{P}(Z > x)$ lorsque $x \to \infty$, pour une solution $Z$ de moyenne finie. Dans les cadres du processus de Galton–Watson (où $A_i = 1/\mathbb{E}N \in ]0, 1[$) et du processus de Crump–Mode–Jirina (où $A_i \leqslant 1$), Bingham et Doney [1,2] ont prouvé que quand $\alpha > 1$ n'est pas un entier, $\mathbb{P}(Z > x)$ est une fonction régulière d'ordre $\alpha$ si et seulement s'il en est de même pour $\mathbb{P}(\sum_{i=1}^{N} A_i > x)$ ; de Meyer [6] a montré que le résultat de Bingham et Doney [1] reste valable quand $\alpha > 1$ est un entier. Leur argument était fondé sur l'équation (0.2) et un théorème Tauberien sur la transformée de Laplace. Récemment, la suffisance a été étendue par Iksanov et Polotskiy [7] au cas des marches aléatoires branchantes en utilisant un argument de martingale. L'argument de Bingham et Doney [1,2] utilise fortement la condition que les $A_i$ sont bornés, tandis que l'argument de Iksanov et Polotskiy [7] ne montre pas la nécessité. Nous allons montrer que les résultats de Bingham et Doney [1,2] et de Meyer [6] restent vrais dans le cas général, de sorte qu'ils peuvent être appliqués aux martingales de Mandelbrot et aux marches aléatoires branchantes. Nos résultats améliorent ceux d'Iksanov et Polotskiy [7] au sens que nous montrons une condition nécessaire et suffisante, pas seulement une condition suffisante, pour la variation régulière de $\mathbb{P}(Z > x)$.

Posons pour $x \geqslant 0$,

$$\rho(x) = \mathbb{E}\sum_{i=1}^{N} A_i^x \quad \text{et} \quad \rho'(x) = \mathbb{E}\sum_{i=1}^{N} A_i^x \ln A_i \tag{0.3}$$

si l'espérance existe dans $[-\infty, \infty]$, $\rho'(x)$ étant une expression formelle qui coîncide avec la dérivée de $\rho$ en $x$ lorsque $\rho$ est finie dans un voisinage de $x$. Par Lyons [8], on voit que lorsque $\mathbb{E}\sum_{i=1}^{N} A_i \ln^+ A_i < \infty$ ($\ln^+ x = \max(\ln x, 0)$), alors l'équation (0.1) a une solution $Z \geqslant 0$ avec $\mathbb{E}Z = 1$ si et seulement si $\mathbb{E}Y_1 \ln^+ Y_1 < \infty$ et $\rho'(1) < 0$ ; par Liu [9], on sait que pour $\alpha > 1$, $\mathbb{E}Z^\alpha < \infty$ si et seulement si $\mathbb{E}Y_1^\alpha < \infty$ et $\rho(\alpha) < 1$. Soit

$$R_0 = \left\{\ell : [0, \infty) \to [0, \infty):\ \ell \text{ est measurables et } \lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1\ \forall \lambda > 0\right\}$$

la classe des fonctions à variation lente en $\infty$. Comme d'habitude, nous écrivons $f(s) \sim g(s)$ si $f(s)/g(s) \to 1$ (quand $x \to 0$ ou $\infty$ selon le contexte), et $f(s) = O(g(s))$, $s \in I$ (resp. $s \to 0$), si $|f(s)| \leqslant C|g(s)|$ pour $s \in I$ (resp. pour $s > 0$ assez petit), où $C > 0$ est une constante ; nous soulignons que $C > 0$ est une constante déterministe lorsque $g$ est une fonction aléatoire.

Dans les théorèmes suivants, nous posons $Y_1 = \sum_{i=1}^{N} A_i$, et nous supposons que $Z \geqslant 0$ est une solution de l'équation (0.1) avec $\mathbb{E}Z = 1$ :

**Théorème 0.1.** *Supposons que $\rho(\alpha) < 1$ pour un certain $\alpha > 1$ et que $\rho(\alpha + \delta) < \infty$ pour un certain $\delta > 0$. Soit $\ell \in R_0$. Alors lorsque $x \to \infty$,*

$$\mathbb{P}(Y_1 > x) \sim x^{-\alpha}\ell(x) \quad \textit{si et seulement si} \quad \mathbb{P}(Z > x) \sim \bigl(1 - \rho(\alpha)\bigr)^{-1} x^{-\alpha}\ell(x). \tag{0.4}$$

Lorsque $\alpha > 1$ n'est pas un entier, le résultat a été démontré par Bingham et Doney [1,2] d'abord pour le processus de Galton–Watson, et ensuite pour le processus de Crump–Mode–Jirina ; lorsque $\alpha > 1$ est un entier, il a été montré par de Meyer [6] pour le processus de Galton–Watson. Dans le cas général, la suffisance a été montré par Iksanov et Polotskiy [7] par des techniques de martingale ; la nécessité est nouvelle. Notre approche utilise des théorèmes Tauberiens comme dans Bingham et Doney [2] et de Meyer [6], et donne une nouvelle démonstration du résultat de Iksanov et Polotskiy [7].

Lorsque $\alpha = 1$, la situation est différente. Dans ce cas nous avons le résultat suivant qui est une extension des résultats correspondants de Bingham et Doney [1,2] pour les processus de Galton–Watson et de Crump–Mode–Jirina :

**Théorème 0.2.** *Supposons que $\rho(1 + \delta) < \infty$ pour un certain $\delta > 0$ et que $\mu := -\rho'(1) \in (0, \infty)$. Soit $\ell \in R_0$ telle que $\int_1^\infty \ell(x)\,\mathrm{d}x/x < \infty$. Si $\mathbb{E}Y_1 \mathbf{1}_{\{Y_1 > x\}} \sim \ell(x)$ $(x \to \infty)$, alors*

$$\mathbb{E}Z\mathbf{1}_{\{Z > x\}} \sim \frac{1}{\mu}\int_x^\infty \ell(t)\,\mathrm{d}t/t \quad (x \to \infty). \tag{0.5}$$

## 1. Introduction and results

Let $(N, A_1, A_2, \ldots)$ be a sequence of random variables with $N \in \mathbb{N} \cup \{\infty\}$ and $A_i \in \mathbb{R}_+$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{R}_+ = [0, \infty)$. Assume that $\mathbb{E}\sum_{i=1}^N A_i = 1$, and consider the distributional equation

$$Z = \sum_{i=1}^N A_i Z_i, \tag{E}$$

where $Z_i$ are nonnegative random variables independent of each other and independent of $(N, A_1, A_2, \ldots)$, each has the same law as $Z$ which is unknown. In terms of the Laplace transform $\phi(s) = \mathbb{E}e^{-sZ}$, Eq. (E) reads

$$\phi(s) = \mathbb{E}\prod_{i=1}^N \phi(sA_i), \quad s \geqslant 0. \tag{E$'$}$$

The study of Eq. (E) is interesting due to a large number of applications in a variety of applied probability settings, including branching processes, self-similar cascades, infinite particles systems, branching random walks, quicksort algorithm and random fractals. Here we study the regular variation of the tail probability $\mathbb{P}(Z > x)$ when $x \to \infty$, for a solution $Z$ of (E) with finite mean. In the context of the Galton–Watson process (where $A_i = 1/\mathbb{E}N \in (0, 1)$) and the Crump–Mode–Jirina process (where $A_i \leqslant 1$), Bingham and Doney [1,2] proved that when $\alpha > 1$ is not an integer, $\mathbb{P}(Z > x)$ is a regular function of order $\alpha$ if and only if the same is true for $\mathbb{P}(\sum_{i=1}^N A_i > x)$; de Meyer [6] showed that the result of Bingham and Doney [1] remains valid when $\alpha > 1$ is an integer. Their argument was based on the functional equation (E$'$) and a powerful Tauberian theorem on Laplace transforms. Recently, the "if" part has been extended by Iksanov and Polotskiy [7] to the case of branching random walks using an elegant martingale argument. The argument of Bingham and Doney [1,2] depends heavily on the boundedness of $A_i$, while the argument of Iksanov and Polotskiy [7] cannot show the "only if" part. We shall show that the results of Bingham and Doney [1,2] and de Meyer [6] remain true in the general case, so that they can be applied to Mandelbrot's martingales and to branching random walks. Our results improve those of Iksanov and Polotskiy [7] in the sense that we show a necessary and sufficient condition, not just a sufficient condition, for the regular variation of $\mathbb{P}(Z > x)$.

For $x \geqslant 0$, set

$$\rho(x) = \mathbb{E}\sum_{i=1}^N A_i^x \quad \text{and} \quad \rho'(x) = \mathbb{E}\sum_{i=1}^N A_i^x \ln A_i \tag{1.1}$$

if it exists in $[-\infty, +\infty]$, where $\rho'(x)$ is a formal expression which coincides with the derivative of $\rho$ at $x$ when $\rho$ is finite in a neighborhood of $x$. By Lyons [8], we see that when $\mathbb{E}\sum_{i=1}^N A_i \ln^+ A_i < \infty$ (where $\ln^+ x = \max(\ln x, 0)$), then Eq. (E) has a solution $Z \geqslant 0$ with $\mathbb{E}Z = 1$ if and only if $\mathbb{E}Y_1 \ln^+ Y_1 < \infty$ and $\rho'(1) < 0$; by Liu [9], we know that for $\alpha > 1$, $\mathbb{E}Z^\alpha < \infty$ if and only if $\mathbb{E}Y_1^\alpha < \infty$ and $\rho(\alpha) < 1$. Let

$$R_0 = \left\{\ell : [0, \infty) \to [0, \infty) \colon \ell \text{ is measurable and } \lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1\ \forall \lambda > 0\right\} \tag{1.2}$$

be the set of functions slowly varying at $\infty$. As usual, we write $f(s) \sim g(s)$ if $f(s)/g(s) \to 1$ (where $s \to 0$ or $\infty$ according to the context), and $f(s) = O(g(s))$, $s \in I$ (resp. $s \to 0$), if $|f(s)| \leqslant C|g(s)|$ for $s \in I$ (resp. for $s > 0$ small enough), where $C > 0$ is a constant; we emphasize that $C > 0$ is a deterministic constant when $g$ is a random function.

In the following theorems, we set $Y_1 = \sum_{i=1}^N A_i$, and assume that $Z \geqslant 0$ is a solution of (E) with $\mathbb{E}Z = 1$:

**Theorem 1.1.** *Assume that $\rho(\alpha) < 1$ for some $\alpha > 1$ and $\rho(\alpha + \delta) < \infty$ for some $\delta > 0$. Let $\ell \in R_0$. Then as $x \to \infty$,*

$$\mathbb{P}(Y_1 > x) \sim x^{-\alpha}\ell(x) \quad \text{if and only if} \quad \mathbb{P}(Z > x) \sim \bigl(1 - \rho(\alpha)\bigr)^{-1} x^{-\alpha}\ell(x). \tag{1.3}$$

When $\alpha > 1$ is not an integer, the result was shown by Bingham and Doney [1,2] first for the Galton–Watson process, and then for the Crump–Mode–Jirina process; when $\alpha > 1$ is an integer, it was shown by de Meyer [6] for the Galton–Watson process. For branching random walks, the implication from $Y_1$ to $Z$ was shown by Iksanov and Polotskiy [7] by martingale techniques; the implication from $Z$ to $Y_1$ is new. Rösler, Topchii and Vatutin [10] showed conditions under which $Z$ belongs to the domain of attraction of a stable law with index $1 < \alpha \leqslant 2$. Our approach uses Tauberian theorems as in Bingham and Doney [2] and de Meyer [6], and gives a new proof of the result of Iksanov and Polotskiy [7].

When $\alpha = 1$, the situation is different. In this case we have the following result which extends the corresponding ones of Bingham and Doney [1,2] for the Galton–Watson process and the Crump–Mode–Jirina process:

**Theorem 1.2.** *Assume that $\rho(1 + \delta) < \infty$ for some $\delta > 0$ and that $\mu := -\rho'(1) \in (0, \infty)$. Let $\ell \in R_0$ be such that $\int_1^\infty \ell(x)\,\mathrm{d}x/x < \infty$. If $\mathbb{E}Y_1 \mathbf{1}_{\{Y_1 > x\}} \sim \ell(x)$ $(x \to \infty)$, then*

$$\mathbb{E}Z\mathbf{1}_{\{Z>x\}} \sim \frac{1}{\mu}\int_x^\infty \ell(t)\,\mathrm{d}t/t \quad (x\to\infty). \tag{1.4}$$

## 2. Key ideas of the approach

The proofs of theorems are mainly based on Tauberian theorems and relations between the Laplace transforms of $Z$ and $Y_1$. For a nonnegative random variable $X$, we write $\phi_X(s)=\mathbb{E}e^{-sX}$ for its Laplace transform and $\mu_{r,X}=\mathbb{E}X^r$ for its $r$-th moment ($r=0,1,\ldots$). If $X$ has at least finite $n$-th moment, we set

$$\phi_{n,X}(s)=(-1)^{n+1}\left\{\phi_X(s)-\sum_{r=0}^{n}\frac{\mu_{r,X}(-s)^r}{r!}\right\},\quad n\geqslant 0. \tag{2.1}$$

Then we have the following two Tauberian theorems which give necessary and sufficient conditions for $\mathbb{P}(X>x)\sim x^{-\alpha}\ell(x)$, as $x\to\infty$: see Proposition 2.1 for $\alpha\notin\mathbb{N}$, and Proposition 2.2 for $\alpha\in\mathbb{N}$:

**Proposition 2.1.** *(See [1, Theorem A].) If $\mu_{r,X}<\infty$ for $r=0,1,\ldots,n$ and $\alpha=n+\beta\geqslant 1$ with $n\in\{1,2,\ldots\}$ and $\beta\in[0,1)$, then the following statements are equivalent for $\ell\in R_0$:*

$$\phi_{n,X}(s)\sim s^\alpha\ell(1/s)\quad (s\to 0);$$

$$\mathbb{E}X^n\mathbf{1}_{\{X>x\}}\sim n!\ell(x)\quad (x\to\infty)\ \textit{when}\ \beta=0,$$

$$\mathbb{P}(X>x)\sim\big[(-1)^n/\Gamma(1-\alpha)\big]x^{-\alpha}\ell(x)\quad (x\to\infty)\ \textit{when}\ 0<\beta<1.$$

**Proposition 2.2.** *Let $n\in\{1,2,\ldots\}$. Assume that $\mu_{n-1,X}<\infty$ and define $\hat\phi_{n-1,X}(s)=\phi_{n-1,X}(s)/s^n$ ($s>0$). Then for $\ell\in R_0$,*

$$\lim_{x\to\infty}\frac{x^n\mathbb{P}(X>x)}{\ell(x)}=1\quad \textit{if and only if}\quad \lim_{t\to 0}\frac{\hat\phi_{n-1,X}(t)-\hat\phi_{n-1,X}(\lambda t)}{\ell(1/t)/(n-1)!}=\ln\lambda\quad \forall\lambda>1. \tag{2.2}$$

Proposition 2.1 was established by Bingham and Doney [1], using Karamata's theorem (cf. [3]). Proposition 2.2 was observed by de Meyer [5] as a consequence of de Haan's theorem (cf. [3, Theorem 3.9.2], [4] or [5]).

Set $t(s)=-\ln\phi_Z(s)$ and $T(s)=\sum_{i=1}^N t(sA_i)$, and let $M$ be a positive random variable whose distribution is determined by

$$\mathbb{E}f(M)=\mathbb{E}\sum_{i=1}^N A_i f(A_i) \tag{2.3}$$

for any positive and measurable function $f$. Then we have the following comparison theorem on the relations between the Laplace transforms of $Z$ and $Y_1$:

**Theorem 2.1.** *Let $Z$ be a solution of* (E) *with $\mathbb{E}Z=1$.*

(i) *Assume that $\rho(\alpha)<1$ for some $\alpha>1$, and that $\rho(\alpha+\delta)<\infty$ for some $\delta>0$. Assume also that $Y_1$ and $Z$ have finite $r$-th moment for all $r\in[1,\alpha)$.*

(ia) *If $\alpha\in(1,\infty)\backslash\{2,3,\ldots\}$, then writing $\alpha=n+\beta$ with $n\in\mathbb{N}^*$ and $\beta\in(0,1)$, we have for $\epsilon>0$ small enough,*

$$\phi_{n,Z}(s)-\mathbb{E}\big\{M^{-1}\phi_{n,Z}(sM)\big\}=\phi_{n,Y_1}(s)+O\big(s^{\alpha+\epsilon}\big),\quad s\geqslant 0. \tag{2.4}$$

(ib) *If $\alpha=n\in\{2,3,\ldots\}$, then for $\epsilon>0$ small enough and some constant $c_{n-1}$,*

$$\phi_{n-1,Z}(s)-\mathbb{E}\big\{M^{-1}\phi_{n-1,Z}(sM)\big\}=\phi_{n-1,Y_1}(s)+\frac{c_{n-1}}{n!}s^n+O\big(s^{n+\epsilon}\big),\quad s\geqslant 0. \tag{2.5}$$

(ii) *Assume that $\rho(1+\delta)<\infty$ for some $\delta>0$. Then for $\epsilon>0$ small enough,*

$$\phi_{1,Z}(s)-\mathbb{E}\big\{M^{-1}\phi_{1,Z}(sM)\big\}=\mathbb{E}\big\{e^{-T(s)}-1+T(s)\big\}+O\big(s^{1+\epsilon}\big),\quad s\geqslant 0. \tag{2.6}$$

We mention that what is important in (2.4)–(2.6) is the validity of the equations for $s\to 0$.

## 3. Sketch of proofs

**Proof of Theorem 1.1.** We distinguish two cases according to $\alpha \in (1,\infty)\backslash\mathbb{N}$ or $\alpha \in \{2,3,\ldots\}$.

(i) **Case** $\alpha \in (1,\infty)\backslash\mathbb{N}$. By Proposition 2.1, it suffices to show that, as $s \to 0$,

$$\phi_{n,Y_1}(s) \sim s^\alpha \ell(1/s) \quad \text{if and only if} \quad \phi_{n,Z}(s) \sim \frac{s^\alpha \ell(1/s)}{1-\rho(\alpha)}. \tag{3.1}$$

If $\phi_{n,Z} \sim \frac{s^\alpha \ell(1/s)}{1-\rho(\alpha)}$ $(s \to 0)$, then

$$\mathbb{E}M^{-1}\phi_{n,Z}(sM) \sim \frac{\rho(\alpha)}{1-\rho(\alpha)} s^\alpha \ell(1/s) \quad (s \to 0). \tag{3.2}$$

Together with (2.4), this implies that $\phi_{n,Y_1}(s) \sim s^\alpha \ell(1/s)$. Conversely, assume that $\phi_{n,Y_1}(s) \sim s^\alpha \ell(1/s)$ $(s \to 0)$. Define $\tilde{\phi}_{n,Z} = \phi_{n,Z}(s)/s^\alpha$, $\tilde{\phi}_Z(s) = \tilde{\phi}_{n,Z}(s) - \mathbb{E}\{M^{\alpha-1}\tilde{\phi}_{n,Z}(sM)\}$. So $\tilde{\phi}_Z(s) = \tilde{\phi}_{n,Z}(s) - \rho(\alpha)\mathbb{E}\tilde{\phi}_{n,Z}(sB)$, where $B \geqslant 0$ is a random variable whose distribution is determined by $\mathbb{E}f(B) = \frac{1}{\rho(\alpha)}\mathbb{E}M^{\alpha-1}f(M)$ for any positive measurable function $f$. Let $\{B_i\}$ be independent copies of $B$. Then by iteration we can prove that

$$\tilde{\phi}_{n,Z}(s) = \sum_{i=0}^{\infty} \rho^i(\alpha)\mathbb{E}\tilde{\phi}_Z(sB_1\cdots B_i). \tag{3.3}$$

By the assumption that $\phi_{n,Y_1}(s) \sim s^\alpha \ell(1/s)$, the equality (2.4) and the fact that $\mathbb{E}B^\delta = \rho(\alpha+\delta)/\rho(\alpha) < \infty$, we have $\tilde{\phi}_Z(s) \sim \ell(1/s)$ and $\mathbb{E}\tilde{\phi}_Z(sB_1\cdots B_i) \sim \ell(1/s)$ as $s \to 0$. Therefore (3.3) yields that $\tilde{\phi}_{n,Z}(s) \sim \frac{\ell(1/s)}{1-\rho(\alpha)}$ $(s \to 0)$.

(ii) **Case** $\alpha \in \{2,3,\ldots\}$. For a random variable $X \geqslant 0$ and $\lambda > 1$, define $\hat{\Phi}_{n-1,X}(s) = \hat{\phi}_{n-1,X}(s) - \hat{\phi}_{n-1,X}(\lambda s)$, where $\hat{\phi}_{n,X}(s) = \phi_{n,X}(s)/s^{\alpha+1}$. Then by (2.5),

$$\hat{\Phi}_{n-1,Z}(s) - \mathbb{E}\{M^{n-1}\hat{\Phi}_{n-1,Z}(sM)\} = \hat{\Phi}_{n-1,Y_1}(s) + O(s^\epsilon), \quad s \geqslant 0. \tag{3.4}$$

As in Case (i), using (3.4) instead of (2.4), we can show that as $s \to 0$,

$$\hat{\Phi}_{n-1,Y_1}(s) \sim \frac{\ln x}{(n-1)!}\ell(1/s) \quad \text{if and only if} \quad \hat{\Phi}_{n-1,Z}(s) \sim \frac{\ln x}{(n-1)!}\big(1-\rho(n)\big)^{-1}\ell(1/s). \tag{3.5}$$

So the conclusion follows from Proposition 2.2. □

**Proof of Theorem 1.2.** By Proposition 2.1 (with $n=1$ and $\beta=0$), it suffices to show that as $s \to 0$, $\phi_{1,Y_1}(s) \sim s\ell(1/s)$ implies

$$\phi_{1,Z}(s) \sim \mu^{-1} s \int_{1/s}^{\infty} \ell(t)\,dt/t. \tag{3.6}$$

Assume that $\phi_{1,Y_1}(s) \sim s\ell(1/s)(s \to 0)$. Notice that if $\ell_1(x) \sim \ell_2(x)$ $(x \to \infty)$, then $\int_x^\infty \ell_1(t)\,dt/t \sim \int_x^\infty \ell_2(t)\,dt/t$ $(x \to \infty)$. Therefore we can suppose that $\ell(1/s) = \phi_{1,Y_1}(s)/s$, $s > 0$. Let $\hat{\phi}_{1,Z}(s) = \phi_{1,Z}(s)/s$, $\hat{\phi}_Z(s) = \hat{\phi}_{1,Z}(s) - \mathbb{E}\hat{\phi}_{1,Z}(sM)$ and $\{M_i\}$ be independent copies of $M$. Then as in (3.3), we have for $s > 0$,

$$\hat{\phi}_{1,Z}(s) = \sum_{n=0}^{\infty} \mathbb{E}\hat{\phi}_Z(sM_1\cdots M_n). \tag{3.7}$$

On the other hand, we can show that for $\epsilon \in (0,1)$ and $s \geqslant 0$,

$$\phi_{1,Y_1}(s) \geqslant \mathbb{E}\{e^{-T(s)} - 1 + T(s)\} = \phi_{1,Y_1}\big((1-\epsilon)s\big) + O\big(s^{1+\delta}\big). \tag{3.8}$$

So by (2.6), we have

$$\frac{\phi_{1,Y_1}(s)}{s} + O\big(s^\epsilon\big) \geqslant \hat{\phi}_Z(s) \geqslant \frac{\phi_{1,Y_1}((1-\epsilon)s)}{s} + O\big(s^\epsilon\big). \tag{3.9}$$

This implies that in (3.7) we can approximately replace $\hat{\phi}_Z(s)$ by $\ell(1/s)$, to prove that when $s \to 0$,

$$\hat{\phi}_{1,Z}(s) \sim \sum_{n=0}^{\infty} \mathbb{E}\ell\left(\frac{1}{sM_1\cdots M_n}\right) \sim \mu^{-1}\int_{1/s}^{\infty} \ell(t)\,dt/t, \tag{3.10}$$

where in the last step we use the condition that $\mathbb{E}M^\delta = \rho(1+\delta) < \infty$. This gives (3.6). □

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