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Differential Geometry

Ricci tensor of slant submanifolds in a quaternion projective space

Tenseur de Ricci d'une sous-variété oblique M dans un espace projectif quaternionien

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A R T I C L E I N F O

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A B S T R A C T

We study slant submanifolds of quaternion Kaehler manifold and in particular of a quaternion projective space. We obtain a sharp estimate of the Ricci tensor of a slant submanifold M in a quaternion projective space $QP^m(4c)$ in terms of the main extrinsic invariant, namely the square mean curvature.

R É S U M É

Nous définissons et étudions des sous-variétés obliques d'une variété Kahlerienne quaternionienne et plus particuliérement d'un espace projectif quaternionien. Nous obtenons une estimation fine du tenseur de Ricci d'une sous-variété oblique M dans un espace projectif quaternionien, en termes de l'invariant extrinsèque principal, à savoir la courbure moyenne quadratique.

1. Introduction

The differential geometry of slant submanifolds has shown an increasing development since B.Y. Chen defined slant immersion in complex geometry as a natural generalization of both holomorphic immersion and totally real immersion (see [1]). Since then many papers appeared on this topic [3]. Recently, B. Sahin [5] studied slant submanifolds of quaternion Kaehler manifolds and gave characterization theorems and examples of slant submanifolds of quaternion Kaehler manifolds. The aim of this Note is to obtain a sharp estimate of the Ricci tensor of a slant submanifold M in a quaternion projective space $QP^m(4c)$ in terms of the main extrinsic invariant, namely the square mean curvature.

The main result of this Note is as follows:

Theorem 1.1. *Let M be an n-dimensional θ-slant submanifold in a quaternion projective space $QP^m(4c)$. Then*

(i) *For each unit vector $X \in T_pM$, the Ricci tensor $S(X)$ satisfies the inequality*

$$S(X) \leqslant \frac{n^2}{4}H^2 + (n-1)c + 9c\cos^2\theta. \tag{1}$$

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(ii) *The equality case of* (1) *holds identically for all unit tangent vectors at* p *if and only if either* p *is a totally geodesic point or* $n = 2$ *and* M *is totally umbilical point.*

2. Preliminaries

Let $\overline{M}^m$ be an m-dimensional Riemannian manifold with metric g. Then $\overline{M}^m$ is called a quaternion Kaehlerian manifold [4] if there exists a 3-dimensional vector bundle V of tensors of type $(1,1)$ with local basis of almost Hermitian structures J_1, J_2, J_3 such that

(i) $J_1J_2 = -J_2J_1 = J_3$, $J_2J_3 = -J_3J_2 = J_1$, $J_3J_1 = -J_1J_3 = J_2$,
(ii) $\overline{\nabla}_X J_a = \sum_{b=1}^{3} Q_{ab}(X)J_b$, $a = 1,2,3$ for all vector fields X tangent to $\overline{M}$, where $\overline{\nabla}$ denotes the Riemannian connection in $\overline{M}$ and Q_{ab} are certain 1-forms locally defined on $\overline{M}$ such that $Q_{ab} + Q_{ba} = 0$.

A quaternion projective space, denoted by $QP^m(4c)$, is a quaternion Kaehlerian manifold of constant quaternionic sectional curvature $4c$ and its curvature tensor is given by [6]

$$\begin{aligned}\overline{R}(X,Y)Z = c\big\{&g(Y,Z)X - g(X,Z)Y + g(IY,Z)IX - g(IX,Z)IY + 2g(X,IY)IZ + g(JY,Z)JX\\ &- g(JX,Z)JY + 2g(X,JY)JZ + g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ\big\},\end{aligned} \tag{2}$$

for vectors X, Y, Z tangent to QP^m.

Definition. (See [5].) Let M be a submanifold of a quaternion Kaehler manifold $\overline{M}$. Then we say that M is a slant submanifold if for each non-zero vector X tangent to M at p, the angle $\theta(X)$ between J_aX and T_pM $(a = 1,2,3)$ is constant, i.e. it does not depend on choice of $p \in M$ and $X \in T_pM$.

In this case, M is usually named a θ-slant submanifold. In particular, it is obvious that both quaternion submanifolds and totally real submanifolds are slant submanifolds corresponding to $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.

We denote by H the mean curvature vector, that is [2]

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i). \tag{3}$$

Also, we set

$$h_{ij}^r = g\big(h(e_i,e_j),e_r\big), \qquad \|h\|^2 = \sum_{i,j=1}^{n} g\big(h(e_i,e_j),h(e_i,e_j)\big). \tag{4}$$

3. Ricci tensor and squared mean curvature

Proof of Theorem 1.1. (i) Let M be a slant submanifold of a quaternion projective space $QP^m(4c)$. Let $p \in M$ and $\{e_1,\dots,e_n\}$ an orthonormal basis of T_pM and $\{e_{n+1},\dots,e_{4m}\}$ an othonormal basis of $T_p^{\perp}M$. The curvature tensor R of M is related to the curvature tensor $\overline{R}$ of $\overline{M}$ by the following Gauss equation

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + g\big(h(X,W),h(Y,Z)\big) - g\big(h(X,Z),h(Y,W)\big). \tag{5}$$

For any X tangent to M, we put

$$J_aX = T_aX + F_aX, \quad a = 1,2,3, \tag{6}$$

where T_aX (resp. F_aX) denotes tangential (resp. normal) component of J_aX.

Then using Eq. (6) in Gauss equation we have

$$S(X,Y) = c\left\{(n-1)g(X,Y) + 3\sum_{a=1}^{3} g(T_aX,T_aY)\right\} + g\big(\operatorname{tr} h, h(X,Y)\big) - g\big(h(Y),h(X)\big). \tag{7}$$

By plugging the θ-slant condition [5]

$$g(T_aX,T_aY) = \cos^2\theta\, g(X,Y) \tag{8}$$

for any pair X, Y of vectors in T_pM and any p in M, $a = 1, 2, 3$, we get

$$\begin{aligned} S(X,Y) &= c\big(n-1+9\cos^2\theta\big)g(X,Y) + g\big(\operatorname{tr}h, h(X,Y)\big) - g\big(h(Y),h(X)\big) \\ &= c\big(n-1+9\cos^2\theta\big)g(X,Y) + \sum_{r=n+1}^{4m}\big(\operatorname{tr}h^{(r)}\big)g\big(h^r(X),Y\big) - g\big(h^{(r)}(X),h^{(r)}(Y)\big) \\ &= c\big(n-1+9\cos^2\theta\big)g(X,Y) + \frac{1}{4}|\operatorname{tr}h|^2 - \sum_{r=n+1}^{4m} g\left(h^{(r)}(X) - \frac{1}{2}\big(\operatorname{tr}h^{(r)}\big)X, h^{(r)}(Y) - \frac{1}{2}\big(\operatorname{tr}h^{(r)}\big)Y\right) \end{aligned} \tag{9}$$

by setting $h^{(r)}(X) = g(h(X), e_r)$, $r = n+1, \dots, 4m$ (each $h^{(r)}$ is then an endomorphism of TM). From this we readily infer the following explicit expression of the Ricci tensor S of M:

$$S = \left(c\big(n-1+9\cos^2\theta\big) + \frac{1}{4}|\operatorname{tr}h|^2\right)\mathrm{Id} - \sum_{r=n+1}^{4m}\left(h^{(r)} - \frac{1}{2}\big(\operatorname{tr}h^{(r)}\big)\mathrm{Id}\right)^2. \tag{10}$$

Parts (i) and (ii) of the main theorem directly follows from the last expression obtained.

The converse of part (ii) is easy to prove. This completes the proof of Theorem 1.1. □

Remark. In particular, similar results can be obtained for quaternion submanifolds and totally real submanifolds by taking $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.

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